Doctoral Dissertation
Doctoral Program in Pure and Applied Mathematics (33^{th}cycle)

Stability and network design of heterogeneous transportation systems

By

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2022

Declaration

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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2022

* This dissertation is presented in partial fulfillment of the requirements for the degree of ***Philosophiæ Diploma*** **(PhD degree)** in **Pure and Applied Mathematics**.

Acknowledgements

I would like to thank my supervisor, prof. Giacomo Como. I thank him for his continuous guidance and support throughout these years. His valuable suggestions and comments have helped a lot for the successful completion of this work. I would also like to thank prof. Asuman Ozdaglar and Francesca Parise, for their support and availability during my visit at MIT. My thanks go to prof. Saraubh Amin and prof. Azaraksh Malekian, who have carefully read and reviewed this dissertation. I thank them for their encouraging comments and for the insightful suggestions that have allowed me to improve the quality of this work. I also thank Gustav Nilsson and Mark Jeeninga for reading my thesis.

I am very grateful to all the members of our research group, and all the students I shared my time with in the last years. Many of them are now good friends: sharing this journey with friends has been an invaluable privilege.

I would like to express my gratitude to my family, especially my mother Elena and my father Filippo, for the encouragement, motivation and support during all these years. I would also like to thank my brother Francesco and my sister Ilaria. My serenity is due to all of them. Particular thanks are due to my girlfriend Eleonora for her inestimable love that enabled me to complete this work, and to my lovely dog Quark, for the company, especially during the lockdown.

I would like to thank my friends from Squadra Corse Ciclofficina Artigiana, especially Fox, whose influence in my life has been invaluable. Special thanks goes to my friends from Florence: although my visits to Florence have become rare, they are my roots. Finally, I would thank Ellie and James, who have been my family in Boston. Without counting on all these people, I would have never achieved this goal.

Abstract

Nowadays, congestion of transportation networks is a primary problem that impacts negatively the quality of life in urban areas. In this thesis we study the behaviour of transportation networks, whereby users' choices are modelled in a game-theoretic framework. We include in our model the heterogeneity of the users, i.e., we consider that the users may have different preferences on the routes. The heterogeneity is needed to describe many applications of interest, e.g., users informed by different routing apps, or users that trade-off differently time and money. The corresponding game-theoretic models are known as *heterogeneous routing games*.

In the first part of the thesis we investigate the stability of users' equilibria in heterogeneous routing games under evolutionary dynamics, which model how the users revise dynamically their decision in time. We focus specifically on the logit dynamics, where users update their actions with the aim of choosing optimal routes, though suboptimality is sometimes reached due to the presence of noise. We provide sufficient conditions on the network topology under which convergence to a stable equilibrium is guaranteed and, when such conditions are not met, we find sufficient conditions on the properties of the equilibria of the game under which the considered equilibria are asymptotically stable under the logit dynamics. We furthermore characterize the fixed points of the logit dynamics in heterogeneous routing games both in the large and in the vanishing noise regimes. Besides the theoretical interest, stability under evolutionary dynamics is relevant for control applications, e.g., for a planner that needs to know in advance which equilibria are stable and will be reached by evolutionary dynamics, in order to maximize the efficiency of that particular equilibria.

In the second part of the thesis we focus on network design problems (NDPs). Specifically, we consider a NDP where the planner can improve one link in the transportation network, with the goal of minimizing the total travel time experienced

on the network. The difficulty of the problem lies in its bi-level nature, i.e., it involves a network optimization given the equilibrium flows under that particular network. We show that, under suitable assumptions, the total travel time variation corresponding to intervention on a link can be formulated in terms of electrical quantities on a related resistor network, and exploit this characterization to propose an efficient algorithm that selects the optimal link in approximation. We then study the optimality of such procedure in the limit of infinite networks, and provide sufficient conditions on the network under which the approximation error vanishes asymptotically.

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Nomenclature

B	Node-link incidence matrix
$C(\cdot)$	Total travel time on the transportation network
G	Green's function of a random walk
L	Link-route incidence matrix
P	Transition matrix of a random walk
$P(\cdot,\cdot)$	Parallel composition of two graphs
$S(\cdot,\cdot)$	Series composition of two graphs
T_i^+	Return time on node i
T_i	Hitting time on node i
W_{ij}	Conductance between nodes i and j
Δ	Laplacian of resistor networks
Θ	Interaction kernels of evolutionary dynamics
d	Destination of the transportation network
η	Inverse of noise level in logit dynamics
γ_i	Lagrangian multiplier associated to constraint $(Bf)_i = \nu_i$
i_e	Current on link e of a resistor network
κ	Intervention rescale parameter

λ_e Lagrangian multiplier associated to constraint $f_e \geq 0$

$\mathbf{f}$ Network flow distribution

$\mathbf{z}$ Route flow distribution

$\mathcal{E}$ Link set of the transportation network

$\mathcal{G}$ Multigraph describing the transportation network

$\mathcal{L}$ Link set of the resistor network

$\mathcal{N}$ Node set of a graph

$\mathcal{N}_d(\mathcal{K})$ Neighborhood at distance d of a subset of nodes $\mathcal{K}$

$\mathcal{P}$ Population set of heterogeneous routing games

$\mathcal{R}$ Route set of the transportation network

ν Exogenous input to the network

o Origin of the transportation network

$\partial_d(\mathcal{K})$ Nodes at distance d from a node $k \in \mathcal{K}$ and at distance greater than or equal to d from the other nodes in $\mathcal{K}$

τ Throughput on the transportation network

$\theta(e)$ Head of link e

$\xi(e)$ Tail of link e

$c_r(\cdot)$ User cost function of route r

$d_e(\cdot)$ Delay function of link e

r_{ij} Effective resistance between nodes i and j

u Potential distribution over a resistor network

Chapter 1

Introduction

1.1 Motivation

In the last few decades the world has become more and more interconnected, and the movement of people and goods has become pivotal for the proper functioning of human activities and for the economic health of the countries. Due to the increasing demand, many cities are facing the problem of traffic congestion, which leads to increasing levels of pollution and massive waste of time and money [2]. On the other hand, novel technological advancements like Intelligent Transportation Systems (ITS), and the recent development in sensing and communicating technologies, improved the possibilities for feedback control of traffic flows, e.g., by traffic light timing control, ramp-metering and variable speed limits. For these reasons, the analysis, design and control of transportation networks have received an increasing attention in the last years.

A fundamental aspect in traffic modelling is concerned with drivers' route choices. In particular, the advent of routing apps, like Google Maps and Waze, has recently reshaped the habits of transportation networks' users. While in the past users could not know the state of the roads in advance, routing apps provide information on the state of the network in real time, and allow the users to choose optimal routes given the current congestion of the network. Given the increasing awareness on the state of the network, a game-theoretic approach appears suitable to model users' choices in transportation networks. In this representation, users are depicted as selfish decision-makers, routes as strategies, and the travel time associated to the routes as users costs, which depend on the network flow distribution to take into account congestion effects.

A flow configuration under which no one has incentive in changing route is called *Wardrop equilibrium*, from the name of the person who first modelled users of transportation networks by a game-theoretic approach [3]. However, since Wardrop equilibria arise from an uncoordinated behaviour, they may be inefficient for the system. A celebrated quantity to measure the inefficiency of equilibrium flows is the *price of anarchy*, defined as the ratio between the total travel time experienced on the network under Wardrop equilibrium flows, and the total travel time under socially optimal flows [4–7].

The ultimate goal of control of transportation networks is to reduce congestion. While a central planner cannot enforce directly a network flow distribution, a first approach to reduce congestion consists in influencing the users' choices by incentive-design mechanisms, in such a way to align individual costs to social costs, and make users distribute on the network according to socially optimal configurations. This can be done in practice in many ways, e.g., by imposing monetary tolls that make the users pay for the externalities of their choices [8], or by information design mechanisms [9].

Instead of influencing the users' behaviour, a second approach to reduce congestion consists in intervening directly on the infrastructures, by adding new roads or improving existing ones. This class of problems is known in the literature as *network design problems* (NDPs) [10, 11]. The evaluation of the impact of an intervention on the infrastructure is quite difficult, and must take into account the strategic behaviour of the users of the network. A policy-maker that ignores the difference between socially optimal flows and user optimal flows can incur in catastrophic mistakes, well illustrated by Braess' paradox, which shows that improving the transportation network may in some case lead to a degradation of the performance of the network if the users are selfish and uncoordinated [12].

An important aspect in game-theoretic traffic models is concerned with users' preferences. The Wardrop's model assumes *homogeneity* of the users, in the sense that all the users are assumed to take decisions based on identical cost functions. However, this assumption is quite restrictive when modelling many situations of interest. To this aim, homogeneous models have been subsequently generalized to take into account the heterogeneity of the users' cost functions [13]. Among the applications of heterogeneous models we mention: drivers using different TIS, that take decisions based on different perceived cost functions [14, 15]; when fuel consumption or monetary tolls constitute a non-negligible fraction of the user cost, and users have different trade-offs between time and money [16]. From now on, we refer to *heterogeneous routing games* to denote game-theoretic models that incorporate the heterogeneity of users, in contrast with *homogeneous routing games*, which do not consider this aspect.

Besides the heterogeneity of the users, another key aspect in game-theoretic traffic models is the dynamics of network flows. Indeed, the description of the equilibria made so far is completely static and misses a fundamental question, i.e., whether, in case the network flows are far from the equilibrium, the flows will converge to an equilibrium under evolutionary dynamics or not, and which equilibrium will be reached. The distinction between homogeneous and heterogeneous routing games has several implications on the properties of the game and in turn on the asymptotic behaviour of evolutionary dynamics. While the characterization of most of the evolutionary dynamics in homogeneous routing games is provided in the literature [17], the characterization in heterogeneous routing games is still an open issue.

Besides the theoretical interest, the stability of the equilibria under evolutionary dynamics has remarkable implications in practice, and paves the way for control applications, as illustrated by the following example. Imagine a policy-maker that designs a tolling scheme with the goal of aligning a certain Wardrop equilibrium of the game to a socially optimal flow distribution. If the considered equilibrium is unstable from an evolutionary dynamics perspective, then the equilibrium is not reached from the dynamics and the planner's effort to maximize its efficiency is completely wasted. This example shows how understanding whether the network flows will converge to an equilibrium, and identifying which one will be reached by the dynamics in case of multiple equilibria, are fundamental questions for a central planner.

1.2 Related literature

A large branch of the literature focuses on the modelling of physical aspects of traffic flows. A classical traffic model is [18], where the movement of cars on highways is modelled by flows instead of describing single vehicles. Another popular traffic model is the *cell transmission model* (CTM) [19]. This model splits highways in cells, and models the flow through two cells in terms of the upstream demand and the downstream supply. The model is then generalized to describe networks in [20]. Generalizations of [20] are then provided in the literature, to consider more refined rules for flow transmission between cells [21–24]. Traffic flows control has received large attention in the last years. Among its applications, we mention optimal timing in traffic lights [25–27, 23, 28], ramp-metering and variable speed limits [29–31].

The mentioned papers consider routing choices as given, without taking the game-theoretic aspects of the problem into account. The first game-theoretic model for traffic applications has been proposed by Wardrop in [3]. As already mentioned, the Wardrop's model assumes the homogeneity of the users, which however is too restrictive for many applications of practical interest.

Heterogeneity of the users have been first introduced in [13]. The model assumes the existence of a finite number of populations of users that differ in the cost functions. Heterogeneous routing games have been studied to investigate the role of routing apps in transportation networks [14], how a different knowledge on the available routes can impact the total travel time experienced on the network [32], or to model different trade-offs between time and money [16, 8] when the cost experienced by the users is a combination of the two factors.

Both the models in [3, 13] are non-atomic, i.e., they assume that the number of users is very large and treat the set of the users as a continuum, and assume that the action of a single user has a negligible effect on the cost functions. Both [3, 13] assume the *separability* of cost functions, i.e., the cost associated to a road depends only on the flow on the road itself, and assume that the demand is fixed, i.e., not to travel is not an option for the users. However, a wide variety of routing games with different settings have been proposed in literature: see for instance [33–35] for stochastic games in which the travel time is not deterministic due to incomplete information, [36–38] for games with elastic demand, [39–41] for non-separable games, [42] for non-separable routing games with elastic demand, and [16] for games with infinite set of populations. For a complete overview on routing games we refer to [43].

The first who recognized the difference between system optimal flows and users optimal flows is Pigou [44]. This difference has been subsequently formalized via the notion of price of anarchy, which has been extensively studied in the literature

[4–7]. Among incentive-design mechanisms that aim at minimizing the price of anarchy in routing games we mention road tolling [45, 8, 46–48], information design [49, 50, 9, 51] and lottery rewards [52].

As already discussed, an alternative approach to reduce congestion is to improve the transportation network via network design problems (NDPs). NDPs have been first proposed in [10]. Both *continuous* network design problems [53–55], where the budget can be allocated continuously among the roads, and *discrete* formulations, in which the decision variables include which new roads to build [56], how many lanes to add to existing roads [57], or a mix of those two problems [58], have been considered in the literature, together with *dynamical* formulations [59], and formulations where the optimum is achieved by removing, instead of adding, links to the network, because of Braess' paradox [60, 61]. For comprehensive surveys on the literature on NDP we refer to [62, 11].

Evolutionary game theory has been first formulated in [63] to describe animal behaviour in game-theoretic situations, and then applied more generally to the evolution of strategic choices in game theory [64]. For a complete reference on evolutionary dynamics in population games we refer to [17]. Global stability of equilibria under evolutionary dynamics is established for potential, stable and supermodular games [17, Chapter 7]. While homogeneous routing games are potential games, heterogeneous routing games do not belong to any of the mentioned class of games.

To the best of our knowledge, no theoretical results on the global stability under evolutionary dynamics of Wardrop equilibria of heterogeneous routing games are provided in the literature. The speed of convergence to Wardrop equilibria in homogeneous routing games is studied in [65] for no-regrets dynamics, and in [66] for imitative dynamics, but heterogeneous routing games are not included in the analysis. In [67] the convergence of evolutionary dynamics in homogeneous atomic games is investigated. In [68, 69], the authors propose a multiscale model in which the dynamics of users' choice are intertwined with the physical dynamics on the network, but the users are assumed homogeneous.

1.3 Contributions

In this thesis we focus on two main problems related to game-theoretic traffic models. We study non-atomic games, which assume that the number of agents is large and treat the agents' set as a continuum, in the spirit of population games [17].

In Chapter 3 we investigate the stability of Wardrop equilibria of heterogeneous routing games under evolutionary dynamics, focusing specifically on the logit dynamics, which appears to fit traffic applications better than imitative dynamics. We provide original results that relate the network topology and properties of the Wardrop equilibria to the stability of the considered equilibria, and characterize the asymptotic behaviour of evolutionary dynamics in heterogeneous routing games. Our first result is that for every heterogeneous routing game, the corresponding logit dynamics admits a non-empty compact set of fixed points. We show that the set of fixed points approaches a subset of the Wardrop equilibria of the game (called *limit equilibria*) in the limit of vanishing noise, and prove that every strict equilibrium of the game (i.e., an equilibrium flow under which every population uses a single route and all the other routes are strictly suboptimal) belongs to the set of the limit equilibria.

We then investigate the asymptotic behaviour of evolutionary dynamics. Our second result is that evolutionary dynamics that satisfy certain conditions admit a globally asymptotically stable fixed point if the network is parallel (i.e., it has parallel routes from the origin to the destination) or if it is the series composition of parallel networks. Since the logit dynamics satisfies all the required conditions, we use the first and the second results to prove that the unique fixed point of the logit dynamics approaches the set of Wardrop equilibria as the noise vanishes. Such a result generalizes the stability results in homogeneous routing games to the case of heterogeneous routing games, under a restrictive assumption on the network topology. We then characterize the behaviour of the logit dynamics on arbitrary networks, both in the large and vanishing noise regimes, proving that the dynamics may exhibit a bifurcation as the noise in the logit dynamics varies, and that strict equilibria are locally asymptotically stable in the limit of vanishing noise.

Our second main contribution is on network design problems. We study a special class of NDP, where the planner can improve the delay function of a single link, and assume that the travel time associated to every link is affine in the flow. We formulate the problem in the setting homogeneous games. Our goal is to strike a balance between a model that is simple enough to guarantee tractable analysis, yet rich enough to allow insights for more general classes of NDPs. For this class of NDPs, our first result is that, under a regularity assumption that states that the links that carry positive flow remain unchanged with an intervention, the social cost variation corresponding to an intervention on a particular link may be expressed in terms of electrical quantities on a related resistor network. Specifically, the social cost variation depends on the effective resistance of the link (i.e., between the endpoints of the link). Additionally, we show that such a regularity assumption is satisfied provided that the total incoming flow to the network is large enough and the network is series-parallel, which may be of independent interest.

We then propose a method based on Rayleigh's monotonicity laws to approximate the effective resistance of each link with a number of iterations independent of the network size, thus leading to a significant reduction of complexity of the NDP. We then study the optimality of such procedure and provide sufficient conditions on the network under which the approximation error vanishes asymptotically in the limit of infinite networks.

For the future we aim at extending the results to more a general setting. On evolutionary dynamics, we aim at establishing global stability results on arbitrary network topologies, and investigating other evolutionary dynamics like imitative dynamics. Furthermore, we could incorporate the physical dynamics of traffic flows into our model and study the stability properties of the resulting dynamics. On network design problem, natural directions are to extend the analysis to multiple interventions, and to incorporate the users' heterogeneity in the model.

1.4 Organization of the dissertation

The dissertation is organized as follows. In Chapter 2 we introduce the preliminary notions on transportation networks and routing games that are needed for the dissertation. Specifically, we provide basic notions on graphs in Section 2.2, and define network flow optimization in Section 2.3. We then introduce the framework of population games in Section 2.4, and describe homogeneous routing games and heterogeneous routing games in Sections 2.5 and 2.6.

In Chapter 3 we study the asymptotic behaviour of evolutionary dynamics in heterogeneous routing games. The problem is introduced and motivated in Section 3.1. In Section 3.2 we define evolutionary dynamics, focusing in particular on the logit dynamics. We then discuss the asymptotic behaviour of the logit dynamics in homogeneous routing games in Section 3.3. The following sections are devoted to study the logit dynamics in heterogeneous routing games. In Section 3.4 we show that the fixed points of the logit dynamics approach the set of the Wardrop equilibria of the game in the limit of vanishing noise. In Section 3.5 we study the asymptotic behaviour of evolutionary dynamics on parallel networks and their series composition. In Section 3.6 we characterize the asymptotic behaviour of the logit dynamics on arbitrary networks, both in the large and vanishing noise regimes. In Section 3.7 we discuss conjectures and future research lines. Finally, in Section 3.8 we summarise our contribution. The contents of this chapter are based on the conference publication [70] co-authored with my advisor Giacomo Como, as well as on unpublished material that we plan to submit for journal publication. Numerical simulations reported in this chapter were done in collaboration with Tommaso Toso, master thesist at Politecnico di Torino.

In Chapter 4 we study NDPs in homogeneous routing games. In Section 4.1 the problem is introduced and motivated. In Section 4.2 we define formally the model and formulate the optimization problem. In Section 4.3 we provide the electrical formulation of the NDP. In Section 4.4 we show that the effective resistance of a link can be approximated by performing only local computations, and exploit this fact and our electrical formulation to propose an efficient algorithm that finds an approximated solution of the original NDP. In Section 4.5 we analyse the asymptotic performance of our procedure in the limit of infinite networks. In Section 4.6 we provide numerical simulations on synthetic networks and on real transportation networks. In Section 4.7 we discuss extension of our approach, and summarize the work in Section 4.8. These contributions are based on the submitted journal publication [71], co-authored with Giacomo Como, Asuman Ozdaglar and Francesca Parise. The research was conducted during my 15 months visit at the Laboratory for Information and Decision Systems at Massachusetts Institute of Technology.

1.5 Notation

We let $\mathbf{1}_n$, $\mathbf{I}_n$ denote respectively the dimensional vector of all ones with size $n \times 1$, and the identity matrix $n \times n$. Let $\mathbf{0}_{n\times m}$ denote the matrix of size $n \times m$ containing all zero-elements. When indexes are omitted, the size can be deduced from the context. We let δ_i denote the vector containing 1 in the i-th component, and zero otherwise, with the size deducible from the context. Vectors, matrices, and sets are usually denoted by bold symbols, capital letters, and calligraphic letters, respectively. The symbol $\mathbf{x}^T$ denotes the transpose of $\mathbf{x}$. We let $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{N}$ denote respectively the set of real vectors of size n, non-negative real vectors of size n, and natural numbers. $||x||_+$ denotes the positive part of x, which is 0 if $x \leq 0$, and x if $x > 0$. The symbol $\preccurlyeq$ denotes the element-wise inequality between vectors of same size.

Chapter 2

Transportation network preliminaries

2.1 Introduction

In this chapter we introduce the model of transportation network and routing games, which describe how the users distribute on transportation networks. Specifically, in Section 2.2 we provide basic notions on multigraphs. In Section 2.3 we introduce network flow optimization problems. In Section 2.4 we describe population games framework, that shall be applied in Sections 2.5 and 2.6 to define homogeneous routing games and heterogeneous routing games.

2.2 Notions on multigraphs

We define *multigraph* as pairs $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where:

- $\mathcal{N}$ is the set of nodes, whose cardinality is $\mathrm{N} := |\mathcal{N}|$;
- $\mathcal{E}$ is the set of links, whose cardinality is $\mathrm{E} := |\mathcal{E}|$. Every link $e \in \mathcal{E}$ is endowed with a tail $\xi(e) \in \mathcal{N}$ and a head $\theta(e) \in \mathcal{N}$;

A link e with $\xi(e) = i$, $\theta(e) = j$ has to be meant as a connection from node i to node j. Note that multiple links between a pair of nodes are allowed. A multigraph is called *undirected* if, for every pair of nodes i, j, the number of links e such that $\xi(e) = i, \theta(e) = j$ equals the number of links l such that $\xi(e) = j, \theta(e) = i$. Otherwise, it is called *directed*.

Definition 2.1 (Path). *A path from n_0 to n_m is a sequence of links $(e_0, e_1, ..., e_{m-1})$ such that $\xi(e_0) = n_0, \theta(e_0) = n_1$, $\xi(e_1) = n_1, \theta(e_1) = n_2$, ..., $\xi(e_{m-1}) = n_{m-1}$, $\theta(e_{m-1}) = n_m$ and $n_0 \neq n_1 \neq ... \neq n_m$. When not ambiguous, sometimes we shall refer to a path by indicating the sequence of nodes $(n_0, n_1, \cdots, n_m)$ instead of the sequence of links.*

We now introduce two-terminal multigraphs, which play a key role in our applications.

Definition 2.2 (Two-terminal multigraph). *A two-terminal multigraph is a multigraph endowed with an origin-destination pair* (o, d). *Paths from* o *to* d *are called* routes.

We now introduce the notion of series composition and parallel composition of two-terminal multigraphs. We denote by $\mathcal{R}$ the set of routes of a two-terminal multigraph, with $\mathrm{R} := |\mathcal{R}|$, and let $B \in \mathbb{R}^{\mathcal{N} \times \mathcal{E}}$ denote the *node-link incidence matrix*, with entries

$$B_{nl} = \begin{cases} 1 & \text{if } n = \xi(l) \\ -1 & \text{if } n = \theta(l) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Observe that B is not full-rank, since any column sums to 0. For two-terminal multigraphs we can also define the link-route incidence matrix $L \in \mathbb{R}^{\mathcal{E} \times \mathcal{R}}$, with entries

$$L_{lr} = \begin{cases} 1 & \text{if } l \in r \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Definition 2.3 (Series of two-terminal multigraphs). *Two two-terminal multigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected in series if they have a single common node, which is the destination of $\mathcal{G}_1$ and the origin of $\mathcal{G}_2$. We let $\mathcal{G} = S(\mathcal{G}_1, \mathcal{G}_2)$ denote the series composition of $\mathcal{G}_1$ and $\mathcal{G}_2$.*

Definition 2.4 (Parallel of two-terminal multigraphs). *Two two-terminal multigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected in parallel if they have only the origin and the destination in common. We let $\mathcal{G} = P(\mathcal{G}_1, \mathcal{G}_2)$ denote the parallel composition of $\mathcal{G}_1$ and $\mathcal{G}_2$.*

Notice that $S(\mathcal{G}_1, \mathcal{G}_2)$ and $P(\mathcal{G}_1, \mathcal{G}_2)$ are still two-terminal multigraphs. Also, note that given two multigraphs $\mathcal{G}_1, \mathcal{G}_2$ with routes $\mathcal{R}_1, \mathcal{R}_2$, the route spaces of $S(\mathcal{G}_1, \mathcal{G}_2)$

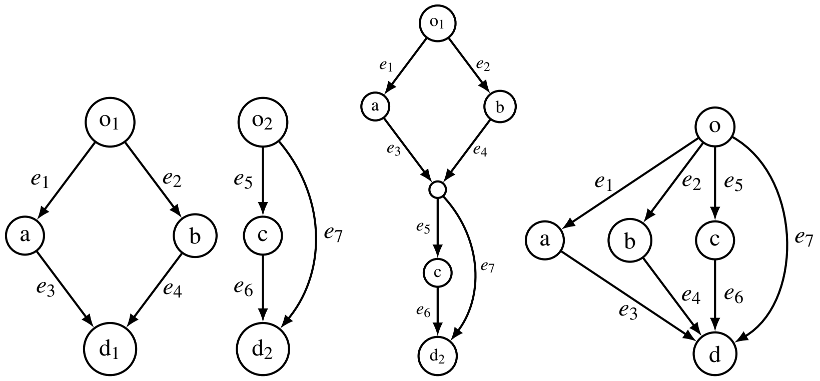


Fig. 2.1 From left to right: two parallel multigraphs, their series composition, and their parallel composition.

and $P(\mathcal{G}_1, \mathcal{G}_2)$ are $\mathcal{R}_1 \times \mathcal{R}_2$ and $\mathcal{R}_1 + \mathcal{R}_2$, respectively. We now introduce parallel multigraphs.

Definition 2.5 (Parallel multigraph)**.** *A two-terminal multigraph is called* parallel *if its routes are parallel, i.e., every link belongs to one route.*

An algebraic characterization of parallel multigraphs can be represented in terms of the link-route matrix L, by saying that a multigraph is parallel if every row of L has one non-zero element.

Example 2.1. *In Figure 2.1 two examples of parallel multigraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are illustrated, together with their parallel and series composition. The route sets of $\mathcal{G}_1$ and $\mathcal{G}_2$ are respectively $\mathcal{R}_1 = \{r_1 = (e_1, e_3), r_2 = (e_2, e_4)\}$ and $\mathcal{R}_2 = \{r_3 = (e_5, e_6), r_4 = e_7\}$. Note that both the multigraphs are parallel. The routes of $S(\mathcal{G}_1, \mathcal{G}_2)$ are*

$$\mathcal{R}_1 \times \mathcal{R}_2 = (r_{13}, r_{14}, r_{23}, r_{24}),$$

where r_{ij} denotes the concatenation of route r_i in $\mathcal{G}_1$ and route r_j in $\mathcal{G}_2$. The routes of $P(\mathcal{G}_1, \mathcal{G}_2)$ are

$$\mathcal{R}_1 + \mathcal{R}_2 = (r_1, r_2, r_3, r_4).$$

Also, note that the parallel composition of two parallel multigraphs is still parallel, whereas the series composition of two parallel multigraphs is not parallel, since every link may belong to multiple routes.

Definition 2.6 (Undirected version)**.** *Given a multigraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, its undirected version is the multigraph $\mathcal{G}_U = (\mathcal{N}, \mathcal{E}_U)$ constructed as follows. The node set is the* *same, while the link set $\mathcal{E}_U$ includes $\mathcal{E}$ plus the minimal set of links that makes the multigraph undirected.*

We now define the notion of *nearly parallel multigraphs*, which has been first formalized in [1], and shall be used to characterize uniqueness of Wardrop equilibria in heterogeneous routing games.

Definition 2.7 (Undirected nearly parallel multigraph [1]). *A two-terminal undirected multigraph is nearly parallel if i) it has a single route, or ii) it has two parallel routes, or iii) can be obtained from a multigraph with two parallel routes at most, by adding an arbitrary number of parallel paths with common end-nodes.*

Definition 2.8 (Directed nearly parallel multigraph). *A two-terminal directed multigraph is nearly parallel if its undirected version is nearly parallel.*

We now introduce the notion of series-parallel multigraph. We start introducing the notion of undirected series-parallel multigraph, and then extend the definition to directed multigraphs.

Definition 2.9 (Undirected series-parallel multigraph). *A two-terminal undirected multigraph is series-parallel if (i) it is composed of two nodes* o, d*, and two opposite links joining* o *to* d *and* d *to* o*, respectively; or (ii) it is the result of connecting two undirected series-parallel multigraphs in parallel; or (iii) it is the result of connecting two undirected series-parallel multigraphs in series.*

The next lemma provides an alternative characterization of undirected-series parallel multigraphs.

Lemma 2.1 ([72]). *Given an undirected two-terminal multigraph, the following statements are equivalent:*

1. *the multigraph is series-parallel;*
2. *for every arbitrary pair of opposite links* e_1 *and* e_2 *such that* $\xi(e_1) = \theta(e_2)$*,* $\theta(e_1) = \xi(e_2)$*, at least one between* e_1 *and* e_2 *does not belong to any route;*
3. *given two arbitrary nodes* n_1 *and* n_2*, if* n_1 *precedes* n_2 *in a route, then* n_1 *precedes* n_2 *in every route.*

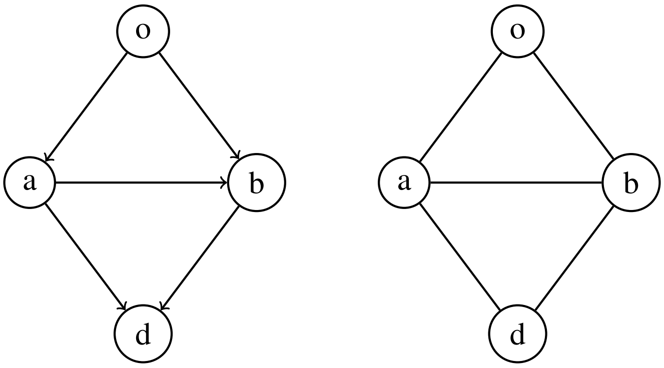


Fig. 2.2 The Wheatstone multigraph and its undirected version. Missing arrows indicate the presence of both a link and its opposite one.

A consequence of Lemma 2.1 is that in undirected series-parallel multigraphs every link has a natural orientation despite the multigraph being undirected, in the sense that only one of the two opposite links may be traversed by a user that travels from the origin to the destination without passing twice from any node. This leads to the definition of directed series-parallel multigraphs.

Definition 2.10 (Directed series-parallel multigraph)**.** *A two-terminal directed multigraph is series-parallel if (i) its undirected version is a series parallel multigraph, and (ii) if it is obtained from its undirected version by removing all the links that do not belong to the route set.*

Example 2.2. *In Figure 2.2 the Wheatstone multigraph and its undirected version are illustrated, where links in undirected multigraphs are graphically represented without arrows. Note that the Wheatstone multigraph is not series-parallel, since in its undirected version is not. Indeed, in the route* (o,a,b,d) *the node a precedes node b, while in the route* (o,b,a,d) *node b precedes node a.*

2.3 Network flow optimization

In this section we present separable convex network flow optimization. Separable convex network flow optimization arises in several applications, e.g., system optimum traffic flows, user optimum network flows in homogeneous routing games, and electrical current flows in resistor networks. Given a multigraph, an *exogenous* *network flow* is a vector $\boldsymbol{\nu} \in \mathrm{R}^{\mathcal{N}}$ such that

$$\sum_{i \in \mathcal{N}} \nu_i = 0. \tag{2.3}$$

A network flow is a vector $\mathbf{f} \in \mathrm{R}^{\mathcal{E}}$ satisfying a positivity constraint and a mass conservation constraints, i.e.,

$$\mathbf{f} \geq \mathbf{0}, \quad B\mathbf{f} = \boldsymbol{\nu}. \tag{2.4}$$

Every link is endowed with a separable non-decreasing convex cost function $\psi_e(f_e)$ such that $\psi_e(0) = 0$. Given an exogenous flow $\boldsymbol{\nu}$ and a network with node-link matrix B, we study the following optimization problem:

$$\mathbf{f}^* \in \operatorname*{arg\,min}_{\substack{\mathbf{f} \in \mathbb{R}_+^{\mathcal{E}} \\ B\mathbf{f} = \boldsymbol{\nu}}} \sum_{e \in \mathcal{E}} \psi_e(f_e). \tag{2.5}$$

The requirement that the cost for sending a flow is non-decreasing in the flow is quite natural, as well as $\psi_e(0) = 0$, which means that there is no cost for sending zero flow. Additionally, convexity of $\psi_e(f_e)$ means that the marginal price for sending the flow is increasing in f_e, i.e., the higher is f_e the higher will be the cost for sending an additional infinitesimal amount of flow. Let $\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{E}}$, $\boldsymbol{\gamma} \in \mathbb{R}^{\mathcal{E}}$ denote the Lagrangian multiplier corresponding to $B\mathbf{f} = \boldsymbol{\nu}$, and $\mathbf{f} \geq \mathbf{0}$, respectively. The next statement establishes necessary conditions for optimality.

Lemma 2.2 ([73])**.** *Consider the optimization problem* (2.5)*. Then:*

1. *A solution* $\mathbf{f}^*$ *exists.*
2. $\mathbf{f}^*$ *is solution of* (2.5) *if there exists a triple* $(\mathbf{f}^*, \boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*)$ *satisfying the conditions*

$$\begin{cases} \psi_e'(f_e^*) + \gamma_{\theta(e)}^* - \gamma_{\xi(e)}^* - \lambda_e^* = 0 & \forall e \in \mathcal{E}, \\ \sum_{\substack{e \in \mathcal{E}: \\ \theta(e)=i}} f_e^* - \sum_{\substack{e \in \mathcal{E}: \\ \xi(e)=i}} f_e^* + \nu_i = 0 & \forall i \in \mathcal{N}, \\ \lambda_e^* f_e^* = 0 & \forall e \in \mathcal{E}, \\ \lambda_e^* \geq 0 & \forall e \in \mathcal{E}, \\ f_e^* \geq 0 & \forall e \in \mathcal{E}. \end{cases} \tag{2.6}$$

3. *if for every link* $\psi_e(f_e)$ *is strictly convex, then* (2.6) *are also sufficient conditions for optimality.*

Eq. (2.6) are known as Karusch-Kuhn-Tucker (KKT). We will not go through the details of optimization theory. For an in-depth reference about convex optimization and optimality conditions we refer to [73]. Note that the optimal flows f_e^* depend on $\boldsymbol{\gamma}^*$ through the difference $\gamma^*_{\xi(e)} - \gamma^*_{\theta(e)}$, implying that the optimal solution is undetermined unless the value of $\boldsymbol{\gamma}^*$ is fixed in an arbitrary node. This is due to the fact that the node-link matrix B is not full rank. The third condition is known as complementary slackness, and implies that all the links such that $\lambda_e^* > 0$ carry a zero flow under the optimal solution, i.e. $f_e^* = 0$. Network flow optimization are used to characterize flows in traffic applications. We introduce system-optimum traffic assignment as an example.

Example 2.3 (System-optimum traffic assignment)**.** *Consider a two-terminal multigraph* $\mathcal{G}$*, and assign to every link a non-decreasing convex delay function* $d_e(f_e)$*. The system-optimum traffic assignment is network flow that minimizes the total travel time, i.e., it is the solution of the network flow optimization*

$$\begin{aligned} \underset{\mathbf{f}}{\operatorname{argmin}} \quad & \sum_{l\in\mathcal{E}} f_l d_l(f_l) \\ \text{subject to} \quad & \mathbf{f} \geq \mathbf{0}, B\mathbf{f} = \nu. \end{aligned} \tag{2.7}$$

2.4 Population games and routing games

The notions introduced so far allows to model transportation networks. This section integrates the previous notions providing tools from game theory, that shall be used to model how the agents interact on the transportation network. Game theory is a standard mathematical framework to model interactions between rational agents that aim at maximizing the utility associated to their action (or equivalently minimizing the user cost). Within this dissertation, we shall use game theory to describe how drivers choose their route in a transportation network based on the congestion level of the roads. In particular, we make use of a specific branch of game theory, known as *population games*, which builds on three main assumptions:

- the number of players is very large;

- a single user choosing a strategy has an infinitesimal effect on users cost functions;
- players interact anonymously, i.e., the user cost associated to each strategy is function of how many users are playing every strategy and not of who is playing what.

In this section, we introduce the basic notions on populations games needed for this dissertation. For a complete reference on population games we refer to the monograph from Sandholm [17]. Population games may be defined by following two different approaches. The first approach is to consider the hydrodynamic limit of anonymous games with a finite number of players, where the word *anonymous* refers to the fact that the user cost associated to the strategies depend on the fraction of players playing every strategy. A second approach is to define population games as an independent instance. We follow the second approach, in line with the cited monograph. For more details on the first approach we refer to [17, Section 11.4]. We are now ready for a formal definition.

Definition 2.11 (Population games). *A population game is a quadruple* $(\mathcal{P},\mathcal{S},\mathbf{c},\boldsymbol{\tau})$*, where*

- $\mathcal{P}=\{1,\cdots,\mathrm{P}\}$ *is the set of populations;*
- $\boldsymbol{\tau}\in\mathbb{R}_+^{\mathcal{P}}$ *is the vector whose element* τ^p *denotes the total mass of population* p.
- $\mathcal{S}=\mathcal{S}^1\times\cdots\times\mathcal{S}^{\mathrm{P}}$ *is the product of strategy sets, where* $\mathcal{S}^p=\{1,\cdots,n^p\}$. *Every player in population* p *selects a strategy in* $\mathcal{S}^p$*, and the aggregate choices of players in that population are collected in the strategy distribution* $\mathbf{z}^p$*, with* $\mathbf{z}^p\in\mathcal{Z}^p=\{\mathbf{z}^p\in\mathbb{R}_+^{\mathcal{S}_p}:\sum_{i\in S^p}z_i^p=\tau^p\}$. *Let the total strategy distribution space be* $\mathcal{Z}:=\mathcal{Z}^1\times\cdots\times\mathcal{Z}^P$*, and* $n:=n^1+\cdots+n^p$*;*
- $\mathbf{c}:\mathcal{Z}\to\mathbb{R}^{\mathcal{S}}$ *is the vector of user cost functions, whose element* $c_i^p:\mathcal{Z}\to\mathbb{R}$ *is the user cost of strategy* i *for a user of population* p*, which depends on the strategy distribution of all the populations.*

We use the term *user cost* to avoid confusion with the cost defined in network flow optimization. Indeed, while user costs refer to cost that a single user aims at minimizing in a game-theoretic framework, the costs in network flows optimizations are to be meant as cost for the whole system. Population games may be distinguished in *homogeneous* games, when all the players are identical ($\mathrm{P}=1$), or *heterogeneous* games ($\mathrm{P}>1$), when players belong to different populations that may differ in the strategy sets and/or in the user cost functions. This distinction may have many implications on the properties of the game, as we shall see in details for routing games in the next sections. An alternative and actually more common definition of population games involves payoff functions instead of user cost functions. We here use the notion of user cost which is more suitable to traffic applications, however the payoff-based formulations may be easily recovered by defining payoffs as the opposite of user cost functions. As in classical game theory, the notion of Nash equilibrium is crucial in population games.

Definition 2.12 (Nash equilibrium)**.** *A* Nash equilibrium *is a strategy distribution* $\mathbf{z} \in \mathcal{Z}$ *such that, for every population* p*, every strategy* $i \in \mathcal{S}^p$ *that is used from population* p *is optimal for that population, i.e., for every* $p \in \mathcal{P}$*,*

$$z_i^p > 0 \implies c_i^p(\mathbf{z}) \leq c_j^p(\mathbf{z}), \quad \forall j \in \mathcal{S}^p. \tag{2.8}$$

In other words, a Nash equilibrium is a strategy distribution such that no one has incentive in changing its strategy, because any alternative strategy has at least the same user cost of the currently used strategy. The definition immediately implies that under a Nash equilibrium all the strategies used by an arbitrary population p share the same user cost. We emphasize that the underlying assumption for this characterization is that the effects of a single individual on user cost functions are negligible. An alternative characterization of Nash equilibria can be formulated in terms of solutions of variational inequalities, as shown in the next proposition.

Proposition 2.1 ([17])**.** *A strategy distribution* $\mathbf{z}$ *is a Nash equilibrium if and only if satisfies*

$$(\mathbf{y}-\mathbf{z})'\mathbf{c}(\mathbf{z}) \geq 0, \quad \forall\, \mathbf{y} \in \mathcal{Z}. \tag{2.9}$$

We now introduce the notion of potential game.

Definition 2.13 (Potential game)**.** *A population game is a* potential game *if there exists a scalar function* $V : \mathcal{Z} \to \mathbb{R}$ *(called* potential*) such that*

$$c_i^p(\mathbf{z}) - c_j^p(\mathbf{z}) = \left(\frac{\partial}{\partial z_i^p} - \frac{\partial}{\partial z_j^p}\right) V(\mathbf{z}). \tag{2.10}$$

The population games framework applied to traffic applications leads to the notion of *non-atomic routing games*. Non-atomic routing games are a particular instance of population games, in which the players represent users of a transportation network, the strategy set is the set of the routes that they can use, and user cost functions are typically the travel time associated to routes, which are non-decreasing functions of the total flow on the links, due to congestion effects. We focus on non-atomic routing games (*routing games* for simplicity), where the word *non-atomic* refers to the fact that a single user has infinitesimal effects on the congestion of the network, coherently with population game framework.

As anticipated in the introduction, we assume that the user costs are *additive*, i.e., the user cost of a route can be expressed as the sum of the user cost of the links that compose that route, and *separable*, i.e., the user cost of each link e is a function of the flow over the link e only. We also assume that the demand does not depend on the congestion of the network, i.e., not to travel is not an option for the users. While additivity appears a natural assumption, and is indeed widely assumed in the literature, separability should be relaxed in many practical cases, e.g., when one wants to consider how flows on two intersecting roads affects latencies of each other. However, all these assumptions are quite standard in the literature, and are made with the purpose of having a model as tractable as possible and are coherent with homogeneous routing games and heterogeneous routing games proposed respectively in [3] and [13]. In the next sections we describe in details routing games, tracing a fundamental distinction between *homogeneous routing games*, in which users make decisions based on the same user cost functions, and *heterogeneous routing games*. As we shall see later, this distinction has several implications on the properties of the game.

2.5 Homogeneous routing games

We model transportation networks as two-terminal multigraphs $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. Since the network is two-terminal, the exogenous flow to the network reads

$$\boldsymbol{\nu} = \tau(\delta_{\text{o}} - \delta_{\text{d}}) \in \mathbb{R}^{\mathcal{N}}, \tag{2.11}$$

where $\tau > 0$ denotes the *throughput* from the origin o to the destination d, equivalent to the notion of mass introduced in the previous section. Since in routing games the strategies correspond to routes, it is natural to express the flows on the network in terms of route flows. Coherently with feasible strategies in population games, a feasible route flow is a vector $\mathbf{z} \in \mathbb{R}^{\mathcal{R}}$ satisfying the non-negativity and conservation of mass constraints, i.e.,

$$\mathbf{z} \geq 0, \quad \mathbf{z}'\mathbf{1} = \tau. \tag{2.12}$$

The route flow induces a unique link flow $\mathbf{f} \in \mathbb{R}_+^{\mathcal{E}}$ via

$$\mathbf{f} = L\mathbf{z}, \tag{2.13}$$

where L is the link-route incidence matrix, i.e., the flow over a link e is the sum of the flow over the routes including e. One can prove by using the flow decomposition theorem [74, Theorem 2.1] that the feasibility of $\mathbf{z}$ is equivalent to requiring that $\mathbf{f} = L\mathbf{z}$ is a network flow, as defined in (2.4), i.e., it satisfies the constraints $\mathbf{f} \geq \mathbf{0}$ and $B\mathbf{f} = \nu$. We assume that every link is endowed with a continuously differentiable delay function $d_e : \mathbb{R}_+ \to \mathbb{R}_+$, which is assumed dependent on f_e and not on the whole $\mathbf{f}$ because of separability assumption. Delay functions are also assumed non-decreasing. The user cost of a route r, under flow distribution $\mathbf{f}$, is additive, i.e., it is the sum of the delays of the links belonging to that route,

$$c_r(\mathbf{f}) = \sum_{l \in \mathcal{E}} L_{lr} d_l(f_l). \tag{2.14}$$

When user cost functions are denoted by $c_r(\mathbf{z})$, we shall implicitly assume that the link flows are the one induced by (2.13). In homogeneous routing games all the users share the set of delay functions, thus the index p for the population may be omitted.

Definition 2.14 (Homogeneous routing game). *A* homogeneous routing game *is a triple* $(\mathcal{G}, \mathbf{d}, \tau)$*, where the transportation network* $\mathcal{G}$ *is a two-terminal multigraph, and* $\mathbf{d}$ *is the set of delay functions of the links.*

We denote routing games by $(\mathcal{G}, \mathbf{d}, \tau)$, in contrast with population games that are defined by $(\mathcal{P}, \mathbf{c}, \tau)$. The omission of $\mathcal{P}$ is motivated by the fact that in homogeneous games it always holds $\mathrm{P} = 1$. Moreover, there exists a unique mapping from $(\mathcal{G}, \mathbf{d})$ to $\mathbf{c}$ via (2.14), which allows to derive from the heterogeneous routing game the corresponding notation for the population game. We prefer to keep the notation

$(\mathcal{G}, \mathbf{d}, \tau)$ in this context because the notion of delay associated to link is standard in the literature of routing games.

The notion of user equilibrium in routing games has been first introduced by Wardrop. A Wardrop equilibrium is a flow distribution such that "the journey times on all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route" [3]. This definition is equivalent to Nash equilibrium in the setting of population games. However, for historical reasons, we refer to Wardrop equilibria to denote the Nash equilibria of non-atomic routing games.

Definition 2.15 (Wardrop equilibrium)**.** *A route flow* $\mathbf{z}^*$ *is a Wardrop equilibrium if for every route r*

$$z_r^* > 0 \implies c_r(\mathbf{z}^*) \leq c_q(\mathbf{z}^*), \quad \forall q \in \mathcal{R}.$$

We refer to $\mathbf{f}^*$ as Wardrop equilibrium if there exists a Wardrop equilibrium $\mathbf{z}^*$ such that $\mathbf{f}^* = L\mathbf{z}^*$. A classical result is that homogeneous routing games are potential games, and the potential function of the game is separable and convex. Hence, Wardrop equilibria are solutions of convex separable optimization problems.

Theorem 2.1 ([36])**.** *A link flow* $\mathbf{f}^*$ *is a Wardrop equilibrium of a homogeneous routing game if and only if it solves the minimization problem*

$$\mathbf{f}^* \in \underset{\substack{\mathbf{f} \in \mathbb{R}_+^{\mathcal{E}} \\ B\mathbf{f} = \nu}}{\arg\min} \sum_{l \in \mathcal{E}} \int_0^{f_l} d_l(s) ds. \tag{2.15}$$

A consequence of Theorem 2.1 is that there is a one-to-one correspondence between Wardrop equilibria of homogeneous routing games and solution of the network flow optimization (2.15). This fact has several remarkable implications that follow from network flow optimization theory (see Section 2.3). First, a Wardrop equilibrium always exists. Second, if the delay functions are assumed strictly increasing, the objective function is strictly convex and the Wardrop equilibrium $\mathbf{f}^*$ is unique. Theorem 2.1 might be enunciated in terms of $\mathbf{z}$ by replacing f_l with $(L\mathbf{z})_l$ in (2.15) and requiring the feasibility of $\mathbf{z}$ instead of $B\mathbf{f} = \nu$. However, the uniqueness of Wardrop equilibrium in terms of route flow distributions does no longer hold. Indeed, in such a formulation, the problem is only convex in $\mathbf{z}$, and thus the game may admit a continuum of equilibrium route flow distributions, all of them inducing equivalent network flow $\mathbf{f}^*$. A third observation that shall be analysed in details in Chapter 3 is that the equilibria are stable under evolutionary dynamics. We now introduce the KKT conditions of problem (2.15), which provide sufficient and necessary conditions under which a flow $\mathbf{f}^*$ is a Wardrop equilibrium. Similar conditions may be established for $\mathbf{z}^*$. For convenience, coherently with network flow optimization framework, we here analyse optimality conditions of (2.15) in terms of link flows $\mathbf{f}^*$. However, with a slight abuse of notation we say that $\mathbf{z}^*$ solves (2.15) if $\mathbf{f}^* = L\mathbf{z}^*$ solves (2.15). We recall that $\boldsymbol{\gamma}$ and $\boldsymbol{\lambda}$ denote the Lagrangian multipliers associated $B\mathbf{f} = \boldsymbol{\nu}$, and $\mathbf{f} \geq 0$, respectively. As shown in Section 2.3, a necessary condition for optimality of $\mathbf{f}^*$ is the existence of a triple $(\mathbf{f}^*, \boldsymbol{\lambda}^*, \boldsymbol{\gamma}^*)$ that satisfies the following KKT conditions:

$$\begin{cases} d_e(f_e^*) + \gamma^*_{\theta(e)} - \gamma^*_{\xi(e)} - \lambda_e^* = 0 & \forall e \in \mathcal{E}, \\ \sum_{\substack{e\in\mathcal{E}:\\ \theta(e)=j}} f_e^* - \sum_{\substack{e\in\mathcal{E}:\\ \xi(e)=j}} f_e^* + \nu_j = 0 & \forall j \in \mathcal{N}, \\ \lambda_e^* f_e^* = 0 & \forall e \in \mathcal{E}, \\ \lambda_e^* \geq 0 & \forall e \in \mathcal{E}, \\ f_e^* \geq 0 & \forall e \in \mathcal{E}. \end{cases} \tag{2.16}$$

Recall that, since B is not full-rank, the equilibrium f_e^* depend on $\boldsymbol{\gamma}^*$ through the difference $\gamma^*_{\theta(e)} - \gamma^*_{\xi(e)}$, implying that (2.16) is undetermined unless the value of $\boldsymbol{\gamma}^*$ is fixed in an arbitrary node. Given a network flow $\mathbf{f}$ (or a route flow distribution $\mathbf{z}$) the total travel time spent on the network is

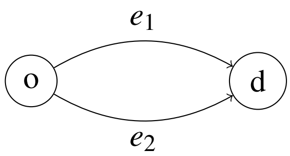
$$C(\mathbf{f}) = \sum_{l\in\mathcal{E}} f_l d_l(f_l). \tag{2.17}$$

From now on, we consider homogeneous routing games with a unique Wardrop equilibrium $\mathbf{f}^*$, and define the social cost as the total travel time at the equilibrium.

Definition 2.16 (Social cost)**.** *Let $\mathbf{f}^*$ be the unique Wardrop equilibrium of a routing game $(\mathcal{G}, \mathbf{d}, \tau)$ with strictly increasing delay functions. The social cost is*

$$C(\mathbf{f}^*) = \sum_{l\in\mathcal{E}} f_l^* d_l(f_l^*).$$

The next proposition shows that the social cost depends only on the difference between the optimal Lagrangian multipliers in o and in d.



$d_1(x)$	$d_2(x)$	τ
x	1	1

Fig. 2.3 A simple game revealing the externality due to uncoordinated behaviour of the users.

Proposition 2.2. *Let* $(\mathcal{G}, \mathbf{d}, \tau)$ *be a routing game. Then,*

$$C(\mathbf{f}^*) = \tau(\gamma_{\text{o}}^* - \gamma_{\text{d}}^*).$$

Proof. We first show that the user cost of every used route, i.e., a route composed of links with positive flows, equals $\gamma_{\text{o}}^* - \gamma_{\text{d}}^*$. Consider a used route $r = (e_1, e_2, ...e_s)$, with $\xi(e_1) = \text{o}$, $\theta(e_s) = \text{d}$, and $\theta(e_i) = \xi(e_{i+1})$ for every $1 \leq i < s$. Thus,

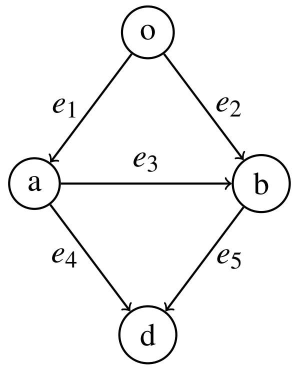
$$c_r(\mathbf{f}^*) = \sum_{i=1}^{s} d_{e_i}(f_{e_i}^*) = \sum_{i=1}^{s} (\gamma_{\xi(e_i)}^* - \gamma_{\theta(e_{i+1})}^* + \lambda_{e_i}^*) = \gamma_{\text{o}}^* - \gamma_{\text{d}}^*.$$

The first equivalence follows from the definition of route user cost, while the second and the third one from KKT conditions, in particular the third one from complementary slackness. Thus, for every $\mathbf{z}^*$ such that $\mathbf{f}^* = L\mathbf{z}^*$,

$$\begin{aligned} C(\mathbf{f}^*) &= \sum_{l \in \mathcal{E}} f_l^* d_l(f_l^*) = \sum_{l \in \mathcal{E}} d_l(f_l^*) \sum_{r \in \mathcal{R}} L_{lr} z_r^* \\ &= \sum_{r \in \mathcal{R}} z_r^* \sum_{l \in \mathcal{E}} L_{lr} d_l(f_l^*) = \sum_{r \in \mathcal{R}} z_r^* c_r(\mathbf{f}^*) = (\gamma_{\text{o}}^* - \gamma_{\text{d}}^*) \sum_{r \in \mathcal{R}} z_r^* = \tau(\gamma_{\text{o}}^* - \gamma_{\text{d}}^*), \end{aligned}$$

concluding the proof. □

Let us denote by $\mathbf{f}^{opt}$ the notion of system-optimum traffic assignment (or optimal flow) introduced in Example 2.3. which is the feasible network flow minimizing the total travel time. The optimal flow $\mathbf{f}^{opt}$ is the flow distribution that a central planner would enforce in the network to minimize the total travel time, in contrast with equilibrium flows, which arise from a selfish uncoordinated behaviour of the users. The ratio between the social cost and the optimal cost is known as *price of anarchy*. We now present two classical examples of homogeneous routing games. The first one is known as Pigou's example, from the name of the person who first recognized the notion of externality and price of anarchy in games [44]. The second example



$d_1(x)$	$d_2(x)$	$d_3(x)$	$d_4(x)$	$d_5(x)$	τ
2x	1	x	1	2x	1/2

Fig. 2.4 A homogeneous game showing that removing a link (e_3 in this case) may lead to an improvement of the total travel time.

is known as Braess' paradox, who first showed that improving the transportation network may impact negatively on the congestion of the network, when the users are selfish and uncoordinated [12].

Example 2.4 ([4])**.** *Consider the homogeneous routing game in Figure 2.3. The unique Wardrop equilibrium is* $f_1^* = 1, f_2^* = 0$. *The social cost is*

$$C(\mathbf{f}^*) = f_1^* \cdot d_1(f_1^*) + f_2^* \cdot d_2(f_2^*) = 1 \cdot 1 + 0 \cdot 1 = 1.$$

The optimal flow is $f_1^{opt} = 1/2, f_2^{opt} = 1/2$, *corresponding to cost* $C(\mathbf{f}^{opt}) = 1/2 \cdot 1/2 + 1/2 = 3/4$. *The price of anarchy is thus* $4/3$.

Example 2.5 (Braess)**.** *Consider the homogeneous routing game in Figure 2.4. By some computations we obtain the Wardrop equilibrium* $f_1^* = f_5^* = 3/8, f_3^* = 1/4, f_2^* = f_4^* = 1/8$. *The corresponding social cost is* $C(\mathbf{f}^*) = 7/8$. *Consider now the network obtained by removing* e_3 *in the previous example. The new equilibrium is* $f_1^* = f_2^* = f_4^* = f_5^* = 1/4$, *and the corresponding social cost is* $C(\mathbf{f}^*) = 3/4 < 7/8$. *Thus, removing link* e_3 *yields a lower social cost.*

The first example shows that the uncoordinated behaviour of the users may lead to a large cost in terms of total travel time experienced on the network, compared to optimal flows. To reduce this effect, an approach proposed in the literature is to influence the behaviour of the users in such a way to align Wardrop flows to optimal flows. A second main approach proposed in the literature to reduce congestion is to intervene directly on the network, by adding new links or improving existing ones. However, the second example shows that in presence of uncoordinated users this must be done carefully, because improving the transportation network may lead to a degradation of the performances of the network. Intervention on transportation network in homogeneous routing games is the subject of Chapter 4. In the next section, we present heterogeneous routing games.

2.6 Heterogeneous routing games

In homogeneous routing games it is assumed that users make decisions based on identical preferences and strategy sets, which is a quite restrictive assumption. In fact, typically the users of a transportation network are not identical, e.g., do not have the same information on the state of the streets, or may travel between different origins and destinations. Heterogeneous routing games are a generalization of homogeneous games whereby users belong to populations that may differ in the origin-destination pair, the route set, and the delay functions. Heterogeneous games have been proposed in [13], and have been widely studied in the literature, e.g. to investigate the value of information provided by routing apps [9, 15, 51] in heterogeneous populations, to model users that have a different knowledge on the available routes [32] or on the state of the streets [75]. Heterogeneous routing games are also considered to model users that trade-off money and time in a different way [16, 8]. Heterogeneous routing games stimulated a lot of interest in the community also from a theoretical perspective. The crucial difference between homogeneous and heterogeneous routing games is that the latter are not in general potential games, which in turn implies that existence, uniqueness, and stability of equilibria under evolutionary dynamics do not trivially follow. However, if users differ only in the origin-destination pairs and the set of available routes, the game still admits a potential function [17, 43], which makes this case less interesting from a theoretical perspective. For this reason, we here consider the case where users differ in the delay functions and for simplicity restrict our analysis to the case of single origin-destination pair for all the populations.

More formally, let the transportation network be a two-terminal multigraph $\mathcal{G}$, and let $\mathcal{P}$ denote the set of populations, whose cardinality $\mathrm{P} = |\mathcal{P}|$ is assumed finite (see e.g. [76, 16] for generalizations to infinite-populations setting, where every single user has personal preferences). We here assume that every population has the same origin-destination pair o-d, and denote by $\tau^p \geq 0$ the throughput that travels from o to d. The populations differ from each other in the assignment of delay functions over the link set. An *admissible route flow distribution* for a population $p \in \mathcal{P}$ is a vector $\mathbf{z}^p \in \mathbb{R}_+^{\mathcal{R}}$ satisfying the throughput constraint, i.e., $\mathbf{1}^T \mathbf{z}^p = \tau^p$. For a given route flow vector $\mathbf{z}^p$, the (unique) link flow distribution is obtained via

$$\mathbf{f}^p = L\mathbf{z}^p, \tag{2.18}$$

Let the *aggregate link flow distribution* and *aggregate route flow distribution* be

$$\mathbf{f}^{agg} = \sum_{p \in \mathcal{P}} \mathbf{f}^p, \quad \mathbf{z}^{agg} = \sum_{p \in \mathcal{P}} \mathbf{z}^p, \tag{2.19}$$

respectively. Let $d_e^p : \mathbb{R}_+ \to \mathbb{R}_+$ be the *delay function* of population $p \in \mathcal{P}$ on the link $e \in \mathcal{E}$. We assume that for every population p and link e the delay function $d_e^p(\cdot)$ depends on the aggregate flow f_e^{agg} over the link itself. We further assume that delay functions are continuously differentiable and non-decreasing in f_e^{agg}. We remark that the populations differ from each other in their delay functions, however the delay on every link depends on the total flow on the link, which looks a reasonable assumption. The user cost of route $r \in \mathcal{R}$ for population p is the sum of delay of links belonging to the route, i.e.,

$$c_r^p(\mathbf{z}) = \sum_{e \in \mathcal{E}} L_{er} d_e^p(f_e^{agg}), \tag{2.20}$$

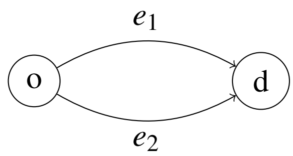
where, given $\mathbf{z}$, the aggregate link flow distribution $\mathbf{f}^{agg}$ is computed by (2.18) and (2.19).

Definition 2.17. *A heterogeneous routing game is a quadruple ($\mathcal{G}$, $\mathcal{P}$, $\mathbf{d}$, $\boldsymbol{\tau}$), where $\mathcal{G}$ is a two-terminal directed multigraph, $\boldsymbol{\tau}$ is the vector of throughputs, and $\mathbf{d}$ denotes the vector containing the delay functions for each link and population.*

The notion of Wardrop equilibrium in heterogeneous game is equivalent to Nash equilibria in heterogeneous population games, and is the natural generalization of homogeneous games.

Definition 2.18 (Wardrop equilibrium)**.** *A Wardrop equilibrium $\mathbf{z}^*$ for the heterogeneous routing game is a feasible route flow such that for every population $p \in \mathcal{P}$, and route $r \in \mathcal{R}$*

$$(z^*)_r^p > 0 \;\Rightarrow\; c_r^p(\mathbf{z}^*) \leq c_q^p(\mathbf{z}^*) \quad \forall q \in \mathcal{R}. \tag{2.21}$$



Pop p	$d_1^p(x)$	$d_2^p(x)$
1	x+1	2x+1
2	x+2	2x

Fig. 2.5 Since the delay functions are in the form $d_e^p(f_e^{agg}) = d_e(f_e^{agg}) + t_e^p$, this routing game admits a potential function despite being heterogeneous.

Thus, at Wardrop equilibrium, no one can unilaterally decrease his user cost by changing route, since every route used by a population p has the minimal user cost (measured by the population p itself) among all the routes. As anticipated, heterogeneous routing games do not admit in general a potential function. The next proposition provides a sufficient and necessary condition under which a heterogeneous routing game admits a potential function.

Proposition 2.3 ([39, 17]). *A heterogeneous game ($\mathcal{G}$, $\mathcal{P}$, $\mathbf{d}, \boldsymbol{\tau}$) is potential if and only if, for every pair of populations p, q and routes i, j,*

$$\sum_{e \in i \cap j} (d_e^p)'(f_e^{agg}) = \sum_{e \in i \cap j} (d_e^q)'(f_e^{agg}). \tag{2.22}$$

A class of heterogeneous games that admit a potential functions are the games with delay functions in the form

$$d_e^p(f_e^{agg}) = d_e(f_e^{agg}) + b_e^p, \tag{2.23}$$

where $d_e(f_e^{agg})$ is a flow-dependent delay that does not depend on the population p and b_e^p is a constant independent of p.

Example 2.6. *Consider a network with two nodes and two parallel links joining* o *to* d, *and a game with* $\mathrm{P} = 2$, *with delay functions as in Figure 2.5. Note that the game admits a potential function, since delay functions are in the form* (2.23). *We now verify that*

$$V(\mathbf{f}) = \frac{(f_1^1 + f_1^2)^2}{2} + f_1^1 + 2f_1^2 + (f_2^1 + f_2^2)^2 + f_2^1,$$

is a potential for the game. Indeed,

$$\begin{cases} \frac{\partial V}{\partial f_1^1} = f_1^1 + f_1^2 + 1 = d_1^1(f_1^{agg}) \\ \frac{\partial V}{\partial f_1^2} = f_1^1 + f_1^2 + 2 = d_1^2(f_1^{agg}) \\ \frac{\partial V}{\partial f_2^1} = 2f_2^1 + 2f_2^2 + 1 = d_2^1(f_2^{agg}) \\ \frac{\partial V}{\partial f_2^2} = 2f_2^1 + 2f_2^2 = d_2^2(f_2^{agg}). \end{cases}$$

By noticing that for the considered network there is a one-to-one correspondence between links and routes, we conclude by (2.10) *that the game is a potential game.*

The last example shows that a potential function may exist even in heterogeneous games. However, this is not usually the case, unless we consider games satisfying restrictive conditions on the delay functions. The existence and uniqueness of equilibria in homogeneous routing games have been proved by using tools from network flow optimization, which cannot be applied to non-potential games. It is natural then to ask whether existence and uniqueness of equilibria still hold even for non-potential heterogeneous games.

2.6.1 Existence and uniqueness of equilibria

Since Wardrop equilibria of heterogeneous games correspond to Nash equilibria of population games, Proposition 2.1 states that Wardrop equilibria correspond the solutions of a variational inequality. This observation is helpful for the characterization of the Wardrop equilibria of heterogeneous games. We point out that such a characterization holds also for more general formulations, e.g., non-separable games, which however are not considered in this dissertation. We now show that every heterogeneous game admits at least a Wardrop equilibria, and that the set of Wardrop equilibria is compact.

Proposition 2.4. *Let $\mathcal{Z}^*$ denote the set of Wardrop equilibria of an arbitrary heterogeneous routing game. Then, $\mathcal{Z}^*$ is non-empty and compact.*

Proof. The existence of at least a Wardrop equilibria is proved in [17, Theorem 2.1.1] by using fixed point techniques. Since $\mathcal{Z}^* \subseteq \mathcal{Z}$, boundedness of $\mathcal{Z}^*$ follows from boundedness of $\mathcal{Z}$. The closeness of $\mathcal{Z}^*$ follows from the characterization of

Wardrop equilibria in terms of variational inequalities, from continuity of user cost functions **c**, and closeness of $\mathcal{Z}$. □

The existence of at least one Wardrop equilibrium has been also proved by Schmeidler in [76] even in a more general setting, that allows every user to belong to a different population (which is equivalent to consider infinite populations). We additionally mention a conceptually different proof, due to Farokhi et al. [77, Theorem 3.3], which exploits the equivalence between Wardrop equilibria in heterogeneous routing games and Nash equilibria in an related abstract game, which has finite number of players (corresponding to populations) and a continuous strategy set. In contrast with the homogeneous games, in which the equilibrium is always unique under the assumption that delay functions are strictly increasing, uniqueness of equilibrium in terms of network flows is not guaranteed to hold in heterogeneous games, as illustrated in the next example. We first provide the definition of essential uniqueness of Wardrop equilibria.

Definition 2.19 (Essentially unique equilibrium)**.** *Consider a heterogeneous routing game, and let $\mathcal{Z}^*$ the set of its Wardrop equilibria. The equilibrium is said to be essentially unique if for every $\mathbf{z},\mathbf{y} \in \mathcal{Z}^*$ the aggregate network flows $\mathbf{f}^{agg}$ induced by $\mathbf{z}$ and $\mathbf{y}$ are equivalent, i.e.,*

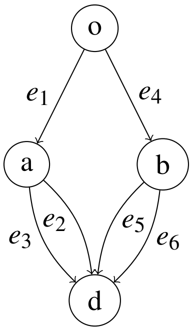
$$\mathbf{f}^{agg} = \sum_{p\in\mathcal{P}} L\mathbf{z}^p = \sum_{p\in\mathcal{P}} L\mathbf{y}^p. \tag{2.24}$$

Example 2.7. *Consider the game in Figure 2.6, and let*

$$r_1 = (e_1,e_2),\ r_2 = (e_1,e_3),\ r_3 = (e_4,e_5),\ r_4 = (e_4,e_6).$$

Note from the delay functions that every population has two available routes. Specifically, population 1 can use routes r_1 and r_4, population 2 can use routes r_1 and r_3, and population 4 can use routes r_2 and r_4. By some computations, we find three Wardrop equilibria, with corresponding aggregate network flows:

1. $$\begin{cases} z_1^1 = 1.2,\ z_4^1 = 0 \\ z_1^2 = 0,\ z_3^2 = 1, \\ z_2^3 = 0,\ z_4^3 = 1. \end{cases} \to \begin{cases} f_1^{agg} = f_2^{agg} = 1.2 \\ f_3^{agg} = 0,\ f_4^{agg} = 2 \\ f_5^{agg} = f_6^{agg} = 1 \end{cases}$$



p	τ^p	d_1	d_2	d_3	d_4	d_5	d_6
1	1.2	19+x	19+x		19+x		19+x
2	1	19+x	20x		19+x	21+x	
3	1	19+x		21+x	19+x		20x

Fig. 2.6 A heterogeneous routing games possessing multiple Wardrop equilibria. A blank in the table means that the delay associated to the link is so high that is never rational to use it.

2. $$\begin{cases} z_1^1 = 0,\ z_4^1 = 1.2 \\ z_1^2 = 1,\ z_3^2 = 0 \\ z_2^3 = 1, z_4^3 = 0. \end{cases} \rightarrow \begin{cases} f_1^{agg} = 2, f_2^{agg} = 1 \\ f_3^{agg} = 1,\ f_4^{agg} = 1.2 \\ f_5^{agg} = 0,\ f_6^{agg} = 1.2 \end{cases}$$

3. $$\begin{cases} z_1^1 = 3/5,\ z_4^1 = 3/5 \\ z_1^2 = 10/21,\ z_3^2 = 11/21 \\ z_2^3 = 11/21,\ z_4^3 = 10/21 \end{cases} \rightarrow \begin{cases} f_1^{agg} = 8/5,\ f_2^{agg} = 113/105 \\ f_3^{agg} = 11/21,\ f_4^{agg} = 8/5 \\ f_5^{agg} = 11,21,\ f_6^{agg} = 113/105 \end{cases}$$

Note that the three equilibria are essentially different, in the sense that they induce different aggregate link flows.

Example 2.7 shows that in general uniqueness and even essential uniqueness are not guaranteed to hold in heterogeneous routing games. Konishi proved in [78] that essential uniqueness holds on series of parallel networks. Milchtaich extended these results, and constructed a beautiful theory on the role of the network topology for uniqueness of the equilibria in two-terminal networks [1]. The next example shows that even if the network is parallel and the equilibrium is essentially unique, i.e., aggregate flows are unique under Wardrop equilibria, a continuum of equilibria in terms of population flows may exist.

Example 2.8. *We consider the multigraph with* $\mathcal{N} = \{\mathrm{o}, \mathrm{d}\}$*, and two parallel links joining* o *to* d*. Consider a routing game with two populations and affine delay functions*

$$d_e^p(f_e^{agg}) = a_e^p f_e^{agg} + b_e^p. \tag{2.25}$$

We assume without loss of generality $b_1^p = 0$ for every population p, since the Wardrop equilibria depend on the difference $b_2^p - b_1^p$ only. We aim at finding conditions under which an internal Wardrop equilibrium exists, i.e., a Wardrop equilibrium such that $(f^*)_e^p > 0$ for every link e and population p. By letting for simplicity of notation $f_e^* := (f^*)_e^{agg}$, such a Wardrop equilibrium satisfies $d_1^p(f_1^*) = d_2^p(f_2^*)$ for both the populations. It follows

$$\begin{cases} a_1^1 f_1^* = a_2^1 f_2^* + b_2^1, \\ a_1^2 f_1^* = a_2^2 f_2^* + b_2^2, \end{cases} \tag{2.26}$$

which admits solution if and only if $a_1^1 a_2^2 - a_1^2 a_2^1 \neq 0$. By some computation, the solution gets:

$$f_1^* = \frac{b_2^1 a_2^2 - b_2^2 a_2^1}{a_2^2 a_1^1 - a_1^1 a_2^1}, \quad f_2^* = \frac{b_2^1 a_1^2 - b_2^2 a_2^1}{a_2^2 a_1^1 - a_1^1 a_2^1}, \tag{2.27}$$

which of course is unique given the uniqueness result from Konishi. An internal equilibrium may exist only if $\tau^1 + \tau^2 = f_1^* + f_2^*$. Additional conditions are that $b_2^1 a_2^2 - b_2^2 a_2^1$, $b_2^1 a_1^2 - b_2^2 a_2^2$, and $a_1^1 a_2^2 - a_1^2 a_2^1$ are either all non-negative or non-positive, to ensure positivity of f_1^* and f_2^*. The population flows at Wardrop equilibrium (we omit the index * for convenience of notation) are

$$\begin{cases} f_1^1 = f_1^1, \\ f_1^2 = f_1^* - f_1^1, \\ f_2^1 = \tau_1 - f_1^1, \\ f_2^2 = f_2^* - \tau_1 + f_1^1, \end{cases} \tag{2.28}$$

showing that the game admits a continuum of equilibria. We now present the uniqueness results from Milchtaich [1]. Such results are originally established for undirected networks, but the directionality of the links may be recovered by assigning infinite delay to links in certain directions.

Definition 2.20 (Uniqueness property)**.** *A multigraph $\mathcal{G}$ is said to have the* uniqueness property *if every heterogeneous game ($\mathcal{G}$, $\mathcal{P}$, $\mathbf{d}, \boldsymbol{\tau}$), admits an essentially unique Wardrop equilibrium.*

Proposition 2.5 ([1])**.** *Every (series composition of) nearly parallel multigraph(s) has the uniqueness property. Moreover, if $\mathcal{G}$ is not a (series composition of) nearly parallel multigraph(s), there always exists a game ($\mathcal{G}$, $\mathcal{P}$, $\mathbf{d}, \boldsymbol{\tau}$) such that the Wardrop equilibrium is not essentially unique.*

In the next chapter we introduce evolutionary dynamics, and investigate the stability of equilibria in heterogeneous routing games under evolutionary dynamics, in relation with the network topology.

Chapter 3

Evolutionary dynamics in routing games

3.1 Introduction

Evolutionary game theory have been first defined by Maynard Smith in [63] to describe animal behaviour in game-theoretic situations, and then applied more generally for the description of the evolution of strategic choices in game theory [64]. So far, the interactions between users have been defined in a static fashion. The notion of Wardrop equilibrium is of large importance in the literature of routing games. The underlying assumption is that equilibrium flows are equivalent to the network flows observed in real applications. However, this assumption is not always justified and requires to be further motivated. In this chapter we investigate stability properties of Wardrop equilibria in heterogeneous routing games under evolutionary dynamics, which model how the users revise their decisions dynamically in time. We focus specifically on the *logit dynamics*, where users update their actions with the aim of choosing optimal routes, though suboptimality is sometimes reached due to the presence of noise. The noise is typically introduced to model uncertainty on the state of the network, or sub-rationality of the users.

From a theoretical perspective, it is known that in homogeneous routing games most of the evolutionary dynamics introduced in the literature converge to the set of the Wardrop equilibria or to perturbations of them [17]. Specifically for the logit dynamics, one can show that the corresponding logit dynamics admits a globally asymptotically stable fixed point, which approaches the set of the Wardrop equilibria of the game as the noise vanishes. However, these results rely on the fact that homogeneous routing games are potential games, and cannot be extended to heterogeneous routing games, since the latter do not admit in general a potential function [41, 77, 17]. As discussed in Chapter 1, many applications of interest require to introduce users' heterogeneity, e.g., to model users that receive information by different routing apps [14, 15], or to model users that have different sensitivity to time and money [8, 16].

To the best of our knowledge, no results on global stability of evolutionary dynamics in heterogeneous routing games are provided in the literature. In most of the literature dealing with heterogeneity of the users in routing games, a big effort is spent to analyse the properties of the equilibria of the game, but the stability of such equilibria is usually not investigated [14, 75, 9, 15, 16]. The speed of convergence of no-regrets dynamics and imitative dynamics are analysed in [65] and [66], but in the considered games the populations differ only in the origin-destination set and not in the delay functions. In the monograph from Sandholm [17], results on the global stability of equilibria in stable, potential or supermodular games are provided, which however do not include the case of heterogeneous routing games.

Our contribution is the following. We characterize the fixed point of the logit dynamics in the limit of vanishing noise, as well as the asymptotic behaviour of evolutionary dynamics, focusing specifically on the logit dynamics. We provide sufficient conditions on the network topology and properties of the evolutionary dynamics under which the dynamics admit a globally asymptotically stable fixed point. Under a suitable assumption on the network topology, our results generalize the global stability under the logit dynamics of Wardrop equilibria of heterogeneous routing games. Additionally, we characterize the logit dynamics on arbitrary networks both in the large noise regime and in the vanishing noise regime.

The chapter is organized as follows. In Section 3.2 we introduce evolutionary dynamics and their properties, focusing specifically on the logit dynamics. In Section 3.3 we analyse the asymptotic behaviour of the logit dynamics in homogeneous routing games. In Section 3.4 we analyse the fixed points of the logit dynamics in heterogeneous routing games. In Section 3.5 we investigate the behaviour of evolutionary dynamics on (series of) parallel networks. In Section 3.6 we study the asymptotic behaviour of the logit dynamics on arbitrary networks. Finally, we discuss conjectures and future research lines in Section 3.7, and summarize the contribution in Section 3.8. The notions on continuous-time dynamical systems needed in this chapter are contained in Appendix B.

For simplicity of notation, we refer to transportation networks by using graph-theoretic notions of the multigraph that models the network, e.g., parallel network, series of composition of networks, etc.

3.2 Evolutionary dynamics

We introduce the notion of evolutionary dynamics for routing games. We adopt from the beginning the notation of routing games, although these notions may be defined for arbitrary population games. *Evolutionary dynamics* are continuous-time dynamical systems, which describe how users revise dynamically their decisions in time. Evolutionary dynamics are fully characterized by the associated interaction kernel. We let $\Theta^p : \mathbb{R}^{\mathcal{R}\times\mathcal{P}} \to \mathbb{R}_+^{\mathcal{R}\times\mathcal{R}\times\mathcal{P}}$ denote the *interaction kernel* of population p, whose element $\Theta^p_{ij}(\mathbf{z})$ indicates the rate at which users of population p who are using route i switch to route j, as a function of the route distribution $\mathbf{z}$. Given interaction kernel, the associated evolutionary dynamics are continuous-time dynamical systems, $(\mathbb{R}^n, g)$, with $g : \mathbb{R}^n \to \mathbb{R}^n$ whose components g^p_i read

$$\dot{z}^p_i = g^p_i(\mathbf{z}) = \sum_{j\in\mathcal{R}} \left(z^p_j \Theta^p_{ji}(\mathbf{z}) - z^p_i \Theta^p_{ij}(\mathbf{z}) \right). \tag{3.1}$$

The first term indicates users using other routes that switch to i, while the second one indicates users that switch to other routes from the route i. Evolutionary dynamics are called

- *target*, if its interaction kernels $\Theta^p_{ab}(\mathbf{z})$ do not depend on a. In such a case, they may be written as
$$\Theta^p_{ab}(\mathbf{z}) \equiv \Theta^p_b(\mathbf{z}); \tag{3.2}$$
- *exact target*, if they are target and for every admissible $\mathbf{z}$ and population p it holds
$$\sum_{b\in\mathcal{R}} \Theta^p_b(\mathbf{z}) = 1; \tag{3.3}$$

In other words, evolutionary dynamics are target if the route adopted by a user after a strategy revision does not depend on the current route, but only on the route distribution $\mathbf{z}$. Exact target dynamics additionally require that the rate at which the users revise their strategy is a constant independent of $\mathbf{z}$ and population p. The next lemma states that exact target dynamics may be written in a simplified form.

Lemma 3.1. *$\mathcal{Z}$ is invariant under exact target evolutionary dynamics. Furthermore, evolutionary dynamics on $\mathcal{Z}$ read*

$$\dot{z}_i^p = \tau^p \Theta_i^p(\mathbf{z}) - z_i^p \tag{3.4}$$

for every population $p \in \mathcal{P}$ and route $i \in \mathcal{R}$.

Proof. Consider $\mathbf{z} \in \mathcal{Z}$, i.e., $\mathbf{1}'\mathbf{z}^p = \tau^p$ for every p. From (3.1)

$$\dot{z}_i^p = \sum_{j\in\mathcal{R}} z_j^p \Theta_{ji}^p(\mathbf{z}) - \sum_{j\in\mathcal{R}} z_i^p \Theta_{ij}^p(\mathbf{z}) \tag{3.5}$$

$$= \sum_{j\in\mathcal{R}} z_j^p \Theta_i^p(\mathbf{z}) - z_i^p \sum_{j\in\mathcal{R}} \Theta_j^p(\mathbf{z}) \tag{3.6}$$

$$= \tau^p \Theta_i^p(\mathbf{z}) - z_i^p, \tag{3.7}$$

where the second equivalence follows from (3.2) and the third one from (3.3) and from $\mathbf{z} \in \mathcal{Z}$. The invariance of $\mathcal{Z}$ follows from

$$\sum_{i\in\mathcal{R}} \dot{z}_i^p = \sum_{i\in\mathcal{R}} \left(\tau^p \Theta_i^p(\mathbf{z}) - z_i^p\right) = \tau^p - \tau^p = 0, \tag{3.8}$$

which concludes the proof. □

From now on we restrict our analysis to exact target evolutionary dynamics, and consider initial conditions belonging to $\mathcal{Z}$. Since $\mathcal{Z}$ is invariant under exact target dynamics, we consider without loss of generality evolutionary dynamics in the form (3.4). We now define two properties to characterize evolutionary dynamics. The first may be defined for every population game, while the second one is specific for routing games. We call evolutionary dynamics:

- *monotone*, if for every pair $\mathbf{z}, \tilde{\mathbf{z}}$, route a and population p satisfying the relation

$$c_a^p(\mathbf{z}) \leq c_a^p(\tilde{\mathbf{z}}),\ c_b^p(\mathbf{z}) \geq c_b^p(\tilde{\mathbf{z}}) \quad \forall b \neq a \tag{3.9}$$

implies that

$$\Theta_a^p(\mathbf{z}) \geq \Theta_a^p(\tilde{\mathbf{z}}). \tag{3.10}$$

- *decoupled* if for any $\mathcal{G} = S(\mathcal{G}_1, \mathcal{G}_2)$, the choices on the two subnetworks are decoupled, i.e., if $r \in \mathcal{R}$ is the route of $\mathcal{G}$ composed of the series of routes $i \in \mathcal{R}_1$ and $j \in \mathcal{R}_2$, then

$$\Theta_r^p(\mathbf{z}) = \Theta_i^p(\mathbf{z}_{\mathcal{G}_1})\Theta_j^p(\mathbf{z}_{\mathcal{G}_2}), \tag{3.11}$$

  where $\mathbf{z}_{\mathcal{G}_1}$ (same arguments apply to $\mathbf{z}_{\mathcal{G}_2}$) is the projection of the route flow distribution on $\mathcal{G}_1$, whose i-th component is

$$z_{i*} := \sum_{j \in \mathcal{R}_2} z_{ij},$$

  and z_{ij} denotes the flow on the route of $\mathcal{G}$ composed of route $i \in \mathcal{R}_1$ in series with route $j \in \mathcal{R}_2$.

A few comments on these properties follow. Under monotonicity assumption, adding some flow to route b makes any parallel route a (i.e., a route not sharing any link with b) more attractive to the users. Monotonicity thus appears suitable for routing games, where users aim at avoiding more congested routes. The decoupling of the choices on series of networks is natural as well. A popular exact target dynamics satisfying the properties defined above is the *logit dynamics*, which arises from the mean-field limit (in the spirit of Kurtz's theorem [79]) of the noisy best response dynamics of classical game theory.

3.2.1 Logit dynamics

The logit dynamics is defined by the interaction kernels

$$\Theta_i^p(\mathbf{z}, \eta) = \frac{\exp(-\eta \cdot c_i^p(\mathbf{z}))}{\sum_{j \in \mathcal{R}} \exp(-\eta \cdot c_j^p(\mathbf{z}))}, \tag{3.12}$$

and reads therefore

$$\dot{z}_i^p = \tau^p \frac{\exp(-\eta \cdot c_i^p(\mathbf{z}))}{\sum_{j \in \mathcal{R}} \exp(-\eta \cdot c_j^p(\mathbf{z}))} - z_i^p, \quad \forall p \in \mathcal{P}, i \in \mathcal{R}, \tag{3.13}$$

where $\eta \in [0,+\infty)$ is the inverse of the *noise level*, and the dependence of the interaction kernel on η is expressed explicitly. We refer to *logit(η)* to denote the continuous-time dynamical system (3.13) for a given value of η. The logit dynamics models users that aim at choosing the optimal strategy, but because of incomplete information or sub-rationality may take suboptimal strategies. The value of η describes how suboptimal the choices of the users are. If $\eta = 0$ (corresponding to infinite noise) the users of every population p choose a strategy in their strategy set with uniform distribution, independently of user cost functions, i.e.,

$$\Theta_i^p(\mathbf{z},0) = \frac{1}{|\mathcal{R}|}. \tag{3.14}$$

As η increases, the noise decreases and the users tend to assign a larger probability to strategies with smaller user cost. Let $\mathcal{R}^p_{min}(\mathbf{z})$ denote the set of optimal routes $i \in \mathcal{R}$ under flow distribution $\mathbf{z}$, i.e., routes such that $c_i^p(\mathbf{z}) \leq c_j^p(\mathbf{z})$ for every other route $j \in \mathcal{R}$. In the limit of infinite η (zero noise), interactions kernels read

$$\lim_{\eta\to+\infty} \Theta_i^p(\mathbf{z},\eta) = \begin{cases} \frac{1}{|\mathcal{R}^p_{min}(\mathbf{z})|}, & \text{if } i \in \mathcal{R}^p_{min}(\mathbf{z}), \\ 0 & \text{otherwise,} \end{cases} \tag{3.15}$$

which means that the users sample uniformly random among the optimal routes. Logit dynamics belongs to the more general class of *perturbed best response* dynamics. The advantage of the logit dynamics compared to other perturbed best response dynamics is that the perturbation level with respect full-rational users is parametrized by the scalar parameter η (see [17, Chapter 6.2] for more details).

Remark 3.1. *Note that the logit dynamics satisfies all the properties defined above: it is target by construction, and exact since for every population p, flow distribution* $\mathbf{z}$*, and* η*, it holds* $\sum_{i\in\mathcal{R}} \Theta_i^p(\mathbf{z},\eta) = 1$*; monotonicity follows from noticing that* $\partial\Theta_i^p/\partial c_i^p \leq 0$*, and* $\partial\Theta_i^p/\partial c_j^p \geq 0$*. To show that it is also decoupled, assume that* $\mathcal{G} = S(\mathcal{G}_1,\mathcal{G}_2)$*, and consider the route r composed of the routes* $i \in \mathcal{R}_1$ *and* $j \in \mathcal{R}_2$ *in series. From additivity of user cost functions, i.e.,* $c_r(\mathbf{z}) = c_i(\mathbf{z}) + c_j(\mathbf{z})$*, it follows*

$$\begin{aligned} \Theta_r^p(\mathbf{z}) &= \tau^p \cdot \frac{\exp(-\eta(c_i^p(\mathbf{z}) + c_j^p(\mathbf{z})))}{\sum_{\substack{n\in\mathcal{R}_1\\ m\in\mathcal{R}_2}} \exp(-\eta(c_n^p(\mathbf{z}) + c_m^p(\mathbf{z})))} \\ &= \tau^p \cdot \frac{\exp(-\eta\cdot c_i^p(\mathbf{z}))}{\sum_{n\in\mathcal{R}_1} \exp(-\eta\cdot c_l^p(\mathbf{z}))} \cdot \frac{\exp(-\eta\cdot c_j^p(\mathbf{z}))}{\sum_{m\in\mathcal{R}_2} \exp(-\eta\cdot c_m^p(\mathbf{z}))}, \end{aligned} \tag{3.16}$$

which satisfies (3.11).

Our main contribution deals with the characterization of the asymptotic behaviour of the logit dynamics in heterogeneous routing games. To better motivate our problem, in the next section we enunciate a known result on logit dynamics in homogeneous routing games. Throughout the chapter we shall assume that no agents use walks from o to d that contain cycles.

3.3 Logit dynamics in homogeneous routing games

In this section we provide a characterization of the asymptotic behaviour of the logit dynamics in homogeneous routing games. Such a characterization relies on the fact that homogeneous routing games are potential games. To this aim, we introduce the entropy $H : \mathcal{Z} \to \mathbb{R}_+$, defined as

$$H(\mathbf{z}) := -\sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}} z_i^p \log\left(\frac{z_i^p}{\tau^p}\right) := \sum_p H_p(\mathbf{z}^p). \tag{3.17}$$

The entropy H_p describes how uniform the strategy distribution of the users of population p is. If all the users of population p use the same strategy, then $H_p = 0$; on the contrary, if the users distribute uniformly on the routes, then H_p is maximized. The next proposition states that the logit dynamics in homogeneous routing games admits a globally asymptotically stable fixed point, and that such a fixed point approaches the set of the Wardrop equilibria of the game as the noise vanishes.

Proposition 3.1. *Consider a homogeneous routing game with potential*

$$V(\mathbf{z}) = \sum_{e\in\mathcal{E}} \int_0^{(L\mathbf{z})_e} d_e(s)ds, \tag{3.18}$$

as defined in (2.15)*, and consider the corresponding logit*(η) *defined in* (3.13)*. Let*

$$V_\eta(\mathbf{z}) := V(\mathbf{z}) - \frac{1}{\eta}H(\mathbf{z}). \tag{3.19}$$

Then:

1. $V_\eta(\mathbf{z})$ *is strictly convex.*

2. *Let $\mathbf{z}_\eta$ denote the unique minimizer of V_η in $\mathcal{Z}$. For every initial condition $\mathbf{z}(0) \in \mathcal{Z}$,*

$$\lim_{t\to+\infty} \mathbf{z}(t) = \mathbf{z}_\eta. \tag{3.20}$$

3. *Let $\mathcal{Z}^*$ denote the set of Wardrop equilibria of the game. Then, $\mathbf{z}_\eta \xrightarrow{\eta\to+\infty} \mathcal{Z}^*$.*

Proof. 1) The strict convexity of V_η follows from noticing that V is convex and $-H$ is strictly convex.

2) The convergence of $\mathbf{z}(t)$ to $\mathbf{z}_\eta$ follows from Point 3 of Proposition C.1 and from the uniqueness of the minimizer $\mathbf{z}_\eta$, which follows from strict convexity of V_η.

3) Consider a sequence η_n such that $\lim_{n\to+\infty}$ and the corresponding sequence of fixed points $\mathbf{z}_n$ is convergent. A convergent sequence always exist due to the fact that $\mathbf{z}_n \in \mathcal{Z}$ and $\mathcal{Z}$ is compact. Denote by $\mathbf{z}^*$ the limit of $\mathbf{z}_n$. It follows that

$$\mathbf{z}^* = \lim_{n\to+\infty} \mathbf{z}_n \in \arg\min \lim_{n\to+\infty} \left(V(\mathbf{z}_n) - \frac{1}{\eta_n} H(\mathbf{z}_n) \right). \tag{3.21}$$

Since H is bounded in $\mathcal{Z}$, and $\lim_{n\to+\infty} \eta_n = +\infty$,

$$\lim_{n\to+\infty} \left(V(\mathbf{z}_n) - \frac{1}{\eta_n} H(\mathbf{z}_n) \right) = V(\mathbf{z}^*) - \lim_{n\to+\infty} \frac{1}{\eta_n} H(\mathbf{z}_n) = V(\mathbf{z}^*). \tag{3.22}$$

Thus, $\mathbf{z}^*$ is a minimizer of V in $\mathcal{Z}$, which implies by Theorem 2.1 that $\mathbf{z}^*$ is a Wardrop equilibrium. □

A few remarks on Proposition 3.1 follow. Note that even if the delay functions are strictly increasing, the potential V is strictly convex in $\mathbf{f}$, but it is in general convex in $\mathbf{z}$. Thus, the game may admit connected set of equilibria $\mathcal{Z}^*$. However, the convexity of $-H$ makes the perturbed potential V_η strictly convex in $\mathbf{z}$, and allows to conclude that for every value of η the logit dynamics admits a globally asymptotically stable fixed point. Proposition 3.1 additionally shows that such a fixed point approaches the set of the Wardrop equilibria of the game as the noise vanishes. Note that Proposition 3.1 does not imply that every Wardrop equilibria $\mathbf{z}^*$ is approached by fixed points of the logit dynamics as η increases. Our main contribution is to extend the results of Proposition 3.1 to the case of heterogeneous routing games. In particular, we prove in Section 3.4 the convergence of the set of fixed points of the logit dynamics to a subset of the Wardrop equilibria in the limit of vanishing noise, called the set of *limit equilibria*, and provide a characterization of such a set. Furthermore, in Section 3.5 we generalize Proposition 3.1 by proving that the logit dynamics admits a globally asymptotically stable fixed point even in heterogeneous routing games, under a suitable assumption on the network topology.

We remark that a result similar to Proposition 3.1 may be established for heterogeneous routing games satisfying condition (2.22), since they admit potential. However, our main effort is on non-potential heterogeneous games, which constitute the largest fraction of heterogeneous games. For completeness of presentation we also remark that despite our main focus is on the logit dynamics, similar results on global convergence on potential games may be established for a broader set of evolutionary dynamics, e.g., imitative dynamics, pairwise comparison dynamics, and excess payoff dynamics (for more details, see [17, Chapter 7.1])

3.4 Fixed points of the logit dynamics

In this section we characterize the fixed points of the logit dynamics in heterogeneous routing games. Specifically, we show that the set of the fixed points of the logit dynamics is non-empty and compact. Additionally, we show that such a set approaches a subset of the Wardrop equilibria (called *limit equilibria*) of the game in the limit of vanishing noise. We furthermore show that every strict Wardrop equilibrium (i.e., an equilibrium in which every population uses one route only, and every other route is strictly suboptimal) belongs to the set of the approximated equilibria. We shall retrieve notions on continuous-time dynamical systems that are contained in Appendix B. Before, we provide the following definitions on Wardrop equilibria.

Definition 3.1 (Strict equilibrium)**.** *An equilibrium* $\mathbf{z}$ *is called* strict *for population* p *if* $\mathbf{z}^p = \tau^p \delta_r$ *for a route* $r \in \mathcal{R}$ *and* $c_r^p(\mathbf{z}) < c_s^p(\mathbf{z})$ *for every* $s \neq r$*. The equilibrium is called* strict *if it is strict for every population* p*.*

Definition 3.2 (Quasistrict equilibrium)**.** *An equilibrium* $\mathbf{z}$ *is called* quasistrict *for population* p *if for every route* r, s *such that* $z_r^p > z_s^p = 0$*, it holds* $c_r^p(\mathbf{z}) < c_s^p(\mathbf{z})$*. The equilibrium is called* quasistrict *if it is quasistrict for every population* p*.*

Theorem 3.1. *Consider a heterogeneous routing game on an arbitrary network* $\mathcal{G}$ *with non-decreasing delay functions, and let* $\mathcal{Z}^* \subseteq \mathcal{Z}$ *denote the set of Wardrop* *equilibria of the game. Let* $\Omega_\eta \subseteq \mathcal{Z}$ *denote the set of fixed points of the associated logit*(η) *defined in* (3.13)*. Then,*

1. *for every* $\eta \geq 0$*,* Ω_η *is non-empty and compact.*
2. *There exists a non-empty compact set* $\overline{\mathcal{Z}^*} \subseteq \mathcal{Z}^*$ *such that*

$$\lim_{\eta \to +\infty} \Omega_\eta = \overline{\mathcal{Z}^*}, \tag{3.23}$$

where the convergence of compact sets is meant in the sense of Definition B.8.

3. *Every strict Wardrop equilibrium belongs to* $\overline{\mathcal{Z}}^*$*.*

Proof. 1) Consider the function $F_\eta : \mathcal{Z} \to \mathcal{Z}$, defined by

$$(F_\eta)_i^p(\mathbf{z}) = \tau^p \frac{\exp(-\eta \cdot c_i^p(\mathbf{z}))}{\sum_{j \in \mathcal{R}} \exp(-\eta \cdot c_j^p(\mathbf{z}))}.$$

Notice that, for every $\eta \in [0,+\infty)$, F_η maps the non-empty compact convex set $\mathcal{Z}$ in itself and is continuous. Hence, Brouwer's fixed point theorem guarantees that F_η admits at least one fixed point in $\mathcal{Z}$ [80]. This implies that the set of fixed points Ω_η is non-empty for every $\eta \in [0,+\infty)$. Notice also that Ω_η is compact for every η, since it is a level set of a continuous function.

2) Consider a sequence $\eta_n \subseteq [0,+\infty)$ such that $\lim_{n\to+\infty} \eta_n = +\infty$. For every n, let $\mathbf{z}_n \in \Omega_{\eta_n}$ be an arbitrary fixed point of logit(η_n). Let $(\mathbf{z}_{n_k})_k$ be a converging subsequence and let $\mathbf{z}^* = \lim_{k\to+\infty} \mathbf{z}_{n_k} \in \mathcal{Z}$ be its limit. Consider a suboptimal route r for population p under $\mathbf{z}^*$, i.e., a route r such that there exists a route s with $c_s^p(\mathbf{z}^*) < c_r^p(\mathbf{z}^*)$. Since route r is dominated by route s, for such a route it holds

$$\lim_{k\to+\infty} \frac{\exp(-\eta_{n_k} \cdot c_r^p(\mathbf{z}_{n_k}))}{\sum_{j \in \mathcal{R}} \exp(-\eta_{n_k} \cdot c_j^p(\mathbf{z}_{n_k}))} = 0. \tag{3.24}$$

From (3.13) and (3.24) it follows

$$(z^*)_r^p = \lim_{k\to+\infty} (z_{n_k})_r^p = \tau^p \cdot \lim_{k\to+\infty} \frac{\exp(-\eta_{n_k} \cdot c_r^p(\mathbf{z}_{n_k}))}{\sum_{j \in \mathcal{R}} \exp(-\eta_{n_k} \cdot c_j^p(\mathbf{z}_{n_k}))} = 0. \tag{3.25}$$

Since this argument may applied to every suboptimal route, only optimal routes may carry a positive flow, which is the definition of Wardrop equilibrium according to Definition 2.17. Thus $\mathbf{z}^*$ belongs to $\mathcal{Z}^*$.

3) Consider a strict equilibrium $\mathbf{z}^*$, and denote by $r^*(p)$ the optimal route for population p, i.e., $(\mathbf{z}^*)^p = \tau^p \delta_{r^*(p)}$ for every p. For any $\varepsilon \geq 0$, let

$$O_\varepsilon = \{\mathbf{z} \in \mathcal{Z} : z^p_{r^*(p)} \geq \tau^p(1-\varepsilon)\ \forall p \in \mathcal{P}\}, \tag{3.26}$$

be the set of route flows such that at least a fraction $1-\varepsilon$ of every population p uses its optimal route $r^*(p)$ under $\mathbf{z}^*$. Note that $\mathbf{z}^* \in O_\varepsilon$ for every $\varepsilon \geq 0$. Let

$$\alpha := \min_{p \in \mathcal{P}} \min_{\substack{s \in \mathcal{R},\\ s \neq r^*(p)}} [c^p_s(\mathbf{z}^*) - c^p_{r^*(p)}(\mathbf{z}^*)] > 0. \tag{3.27}$$

Note that $\alpha > 0$ is a consequence of $\mathbf{z}^*$ strict. We now define $\overline{\varepsilon}$ to be the largest ε such that for every $\mathbf{z} \in O_\varepsilon$, for every population p and route $s \neq r^*(p)$, the difference between the user cost of route s and the user cost of route $r^*(p)$ is at least $\alpha/2$, i.e.,

$$\overline{\varepsilon} = \max\left\{\varepsilon \geq 0 : \min_{\mathbf{z} \in O_\varepsilon} \min_{p \in \mathcal{P}} \min_{\substack{s \in \mathcal{R},\\ s \neq r^*(p)}} [c^p_s(\mathbf{z}) - c^p_{r^*(p)}(\mathbf{z})] \geq \frac{\alpha}{2}\right\}. \tag{3.28}$$

Note that $\overline{\varepsilon} > 0$, since the equilibrium is strict and user cost functions are continuous. We now show the existence of $\overline{\eta}$ such that for every $\eta \in [\overline{\eta}, +\infty)$ and $\varepsilon \in [0, \overline{\varepsilon}]$, F_η maps O_ε in itself. Indeed, for every $\varepsilon \in [0, \overline{\varepsilon}]$ and population p, the route $r^*(p)$ is still strictly optimal for every flow in O_ε. Thus, for an arbitrary $\mathbf{z} \in O_\varepsilon$, route i and population p,

$$\lim_{\eta \to +\infty} (F_\eta)^p_i(\mathbf{z}) = \begin{cases} \tau^p & \text{if } i = r^*(p), \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

Note that the right term in (3.29) correspond to $\mathbf{z}^*$ which is in the interior of O_ε for every $\varepsilon > 0$. Thus, by continuity of F_η in η, there exists a large enough value of $\overline{\eta}$ such that for $\eta \in [\overline{\eta}, +\infty)$, F_η maps O_ε in itself for every $\varepsilon \in (0, \overline{\varepsilon}]$. Since O_ε is compact and convex, Brower's fixed point theorem ensures the existence of at least a fixed point of F_η in O_ε for a large enough η. Since the argument holds for every small enough ε, and since O_ε approximates $\mathbf{z}^*$ as $\varepsilon \to 0$, then there exists a sequence of fixed points $\mathbf{z}_n$ such that $\lim_{n \to +\infty} \mathbf{z}_n = \mathbf{z}^*$, which concludes the proof. □

We call $\overline{\mathcal{Z}^*}$ the set of the *limit equilibria* of the game. Theorem 3.1 states that $\overline{\mathcal{Z}^*}$ includes all the strict equilibria of the game, but a complete characterization of $\overline{\mathcal{Z}^*}$ is still missing. In the next section we generalize Proposition 3.1 to the case of heterogeneous routing game, showing that, under a restrictive assumption on the topology of the network the logit dynamics admits a globally asymptotically stable fixed point.

3.5 Global stability on series of parallel networks

In this section we show that given a heterogeneous game on a (series of) parallel network(s), the corresponding logit dynamics admits a globally asymptotically stable fixed point for every η, and that the set of the fixed points approach the set of the limit equilibria as η grows. The statement builds on contractive systems theory. The steps to the main result are the following:

- we first prove that monotone decoupled exact target evolutionary dynamics whose interactions kernel depends on the flows via the aggregate flows admit a globally asymptotically stable fixed point on (series of) parallel networks;
- since the logit dynamics satisfies all the required properties, we apply the previous statement to the logit dynamics;
- by Theorem 3.1, we conclude that the unique fixed point of the logit dynamics converges to the set of the limit equilibria of the game in the limit of vanishing noise.

The first statement builds on two lemmas and on a technical proposition on contractive systems, whose proof is deferred to Appendix B. Proposition 3.2 is not original and may be found in [81]. Still, we provide an alternative and more intuitive proof. Our proof borrows techniques from [82, Lemma 5], where the authors prove that every monotone diagonally dominant system is l_1-contractive. Proposition 3.2 generalizes this result, proving that l_1-contractivity holds for every continuous-time dynamical system whose Jacobian has strictly negative matrix measure, which includes the monotone case. We refer to Appendix B for notions on continuous-time dynamical systems.

Proposition 3.2. *Let $(\mathbb{R}^n, g)$ be a continuous-time dynamical system, and assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and infinitesimally contractive in $\mathcal{X} \subseteq \mathbb{R}^n$ with respect to $||\cdot||_1$, with contraction rate $c > 0$. Assume that $\mathcal{X}$ is g-invariant, and let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ denote the trajectories at time t corresponding to initial conditions $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}$ and $\mathbf{y}(0) = \mathbf{y}_0 \in \mathcal{X}$, respectively. Then,*

1. *for every $t \geq 0$*
$$||\mathbf{x}(t) - \mathbf{y}(t)||_1 \leq e^{-ct}||\mathbf{x}_0 - \mathbf{y}_0||_1, \tag{3.30}$$
2. *There exists a unique fixed point $\mathbf{z}^*$ in $\mathcal{X}$, and $\mathbf{z}^*$ is globally exponentially stable with region of attraction containing $\mathcal{X}$;*

Proof. See Appendix B. □

The next lemma states that decoupled exact target dynamics on series of networks are equivalent to the dynamics that would be observed on the two subnetworks if multiple games, one on each subnetwork separately, were considered. We consider exact target dynamics
$$\dot{z}_i^p = \tau^p \Theta_i^p(\mathbf{z}) - z_i^p, \tag{3.31}$$
on a network $\mathcal{G} = S(\mathcal{G}_1, \mathcal{G}_2)$. Since the route set of $\mathcal{G}$ is $\mathcal{R}_1 \times \mathcal{R}_2$, we denote every route in $\mathcal{G}$ by two indexes, corresponding to route in $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. We let
$$z_{i*} := \sum_{j \in \mathcal{R}_2} z_{ij}, \quad z_{*j} := \sum_{i \in \mathcal{R}_1} z_{ij}$$
denote the route flows distribution that $\mathbf{z}$ induce on $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

Lemma 3.2 (Decoupling on series of networks)**.** *Let $\mathcal{G} = S(\mathcal{G}_1, \mathcal{G}_2)$, and consider an exact target decoupled dynamics on $\mathcal{G}$. Then, the route flows on $\mathcal{G}_1$ and $\mathcal{G}_2$ induced by the dynamics are equivalent to route flows on $\mathcal{G}_1$ and $\mathcal{G}_2$ induced by separate dynamics on $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.*

Proof. By the assumption that the dynamics is decoupled,
$$\dot{z}_{ij}^p = \tau^p \Theta_i^p(\mathbf{z}_{\mathcal{G}_1}) \Theta_j^p(\mathbf{z}_{\mathcal{G}_2}) - z_{ij}^p. \tag{3.32}$$

Summing (3.32) over j and recalling that the dynamics is exact we get the following dynamics for z_{i*},

$$\dot{z}^p_{i*} = \tau^p \Theta^p_i(\mathbf{z}_{\mathcal{G}_1}) - z^p_{i*}, \tag{3.33}$$

which does not depend on the flows over $\mathcal{G}_2$ and is equivalent to the dynamics that would be observed considering the routing game on $\mathcal{G}_1$ instead of $\mathcal{G}$. The same argument may be applied to $\mathcal{G}_2$, concluding the proof. □

The next lemma states that the convergence of aggregate route flows to a globally asymptotically stable fixed point is a sufficient conditions for the convergence of population flows.

Lemma 3.3. *Consider a heterogeneous routing game, and assume that the aggregate flows* $\mathbf{z}^{agg} = \sum_{p\in\mathcal{P}} \mathbf{z}^p$ *converge to a unique globally asymptotically fixed point under the logit dynamics* (3.13)*. Then, for every population p the route flows* $\mathbf{z}^p$ *converge to a globally asymptotically stable fixed point.*

Proof. We show that convergence of aggregate flows to a globally asymptotically stable fixed point implies the convergence of population flows to a globally asymptotically stable fixed point. Assume that aggregate flows converge asymptotically, and let $\tilde{\mathbf{z}}^{agg} = \lim_{t\to+\infty} \mathbf{z}^{agg}(t)$ denote the fixed point of aggregate flow distribution. Consider the kernels $\Theta(\mathbf{z})$ of the logit dynamics defined in (3.12), where the dependence on η is omitted, and note that, since the user cost functions depend on $\mathbf{z}$ via the aggregate flow $\mathbf{z}^{agg}$, also the kernels depend on $\mathbf{z}$ via the aggregate flow $\mathbf{z}^{agg}$. By convergence of the aggregate flows and by continuity of $\Theta^p_r(\mathbf{z}^{agg})$ for every route and population, we get that for every $\varepsilon > 0$, there exists $T > 0$ such that for every route r and population p,

$$\begin{aligned} |\Theta^p_r(\mathbf{z}^{agg}(t)) - \Theta^p_r(\tilde{\mathbf{z}}^{agg})| < \varepsilon \quad \forall t > T, \\ \Theta^p_r(\tilde{\mathbf{z}}^{agg}) - \varepsilon < \Theta^p_r(\mathbf{z}^{agg}(t)) < \Theta^p_r(\tilde{\mathbf{z}}^{agg}) + \varepsilon \quad \forall t > T. \end{aligned} \tag{3.34}$$

Recall that $\dot{z}^p_r = \Theta^p_r(\mathbf{z}^{agg}) - z^p_r$. Thus, we can use (3.34) to bound $z^p_r(t)$ for every time $t \geq T$. Consider an arbitrary initial condition $z^p_r(T)$. We now prove that for every time $t \geq T$,

$$z^p_r(t) \geq (z^p_r(T) - \Theta^p_r(\tilde{\mathbf{z}}^{agg}) + \varepsilon)e^{-(t-T)} + \Theta^p_r(\tilde{\mathbf{z}}^{agg}) - \varepsilon. \tag{3.35}$$

where the right term of (3.35) is solution of

$$\dot{z}_r^p = \Theta_r^p(\tilde{\mathbf{z}}^{agg}) - \varepsilon - z_r^p. \tag{3.36}$$

with initial condition $z_r^p(T)$. Indeed, let us assume that (3.35) does not hold, i.e., it exists a time $t_2 > T$ such that

$$z_r^p(t_2) < (z_r^p(T) - \Theta_r^p(\tilde{\mathbf{z}}^{agg}) + \varepsilon)e^{-(t_2-T)} + \Theta_e^p(\tilde{\mathbf{z}}^{agg}) - \varepsilon. \tag{3.37}$$

Then, it must exist a time $t_1 \in [T, t_2)$ such that

$$z_r^p(t_1) = (z_r^p(T) - \Theta_r^p(\tilde{\mathbf{z}}^{agg}) + \varepsilon)e^{-(t_1-T)} + \Theta_r^p(\tilde{\mathbf{z}}^{agg}) - \varepsilon, \tag{3.38}$$

$$\dot{z}_r^p(t_1) < \Theta_r^p(\tilde{\mathbf{z}}^{agg}) - \varepsilon - z_r^p(t_1). \tag{3.39}$$

This would imply $\Theta_r^p(\mathbf{z}^{agg}(t_1)) - z_r^p(t_1) < \Theta_r^p(\tilde{\mathbf{z}}^{agg}) - \varepsilon - z_r^p(t_1)$, contradicting (3.34). Thus, from (3.35), in the limit of infinite t,

$$\lim_{t\to+\infty} z_r^p(t) \geq \Theta_r^p(\tilde{\mathbf{z}}^{agg}) - \varepsilon. \tag{3.40}$$

By applying the same arguments to the opposite inequality in (3.34), we obtain

$$\lim_{t\to+\infty} z_r^p(t) \leq \Theta_r^p(\tilde{\mathbf{z}}^{agg}) + \varepsilon. \tag{3.41}$$

Since ε can be chosen arbitrarily small,

$$\lim_{t\to+\infty} z_r^p(t) = \Theta_r^p(\tilde{\mathbf{z}}^{agg}) \quad \forall r \in \mathcal{R},\ \forall p \in \mathcal{P}. \tag{3.42}$$

Hence, $z_r^p(t)$ converges to a unique fixed point for each initial condition, which concludes the proof. □

We can now state our result on the existence of a globally asymptotically stable fixed point of monotone decoupled exact target dynamics for heterogeneous routing games on (series of) parallel networks.

Proposition 3.3. *Consider a heterogeneous game ($\mathcal{G}$, $\mathcal{P}$, $\mathbf{d}, \boldsymbol{\tau}$), and consider an exact target monotone dynamics for this game. Assume additionally that the interaction kernels of the dynamics depend on $\mathbf{z}$ via the aggregate flows $\mathbf{z}^{agg}$, i.e., $\Theta_r^p(\mathbf{z}) \equiv \Theta_r^p(\mathbf{z}^{agg})$ for every route and population. Then:*

1. *If $\mathcal{G}$ is a parallel network, then the dynamics admits a globally asymptotically stable fixed point, and in particular the aggregate flows $\mathbf{z}^{agg}$ converge exponentially fast.*

2. *If $\mathcal{G}$ is a series of parallel networks and the dynamics is also decoupled, then the dynamics admits a globally asymptotically stable fixed point.*

Remark 3.2. *Notice that series of parallel networks differ from series-parallel networks. While the former networks are obtained by composing in series an arbitrary number of parallel networks, the latter ones may be obtained by applying an arbitrary number of series or parallel compositions in any order, and are therefore a superset of the former ones.*

Proof. Summing (3.31) over the populations p, we obtain the dynamics of aggregate route flows, i.e.,

$$\dot{z}_i^{agg} = \sum_{p \in \mathcal{P}} \tau^p \Theta_i^p(\mathbf{z}^{agg}) - z_i^{agg}. \tag{3.43}$$

Since the interaction kernels depend on the aggregate flows, the evolution of the aggregate flows is autonomous. Also, the Jacobian of the system is Metzler (see Definition B.11) because of monotonicity assumption. Furthermore,

$$\sum_{i \in \mathcal{R}} \dot{z}_i^{agg} = \sum_{p \in \mathcal{P}} \tau^p \sum_{i \in \mathcal{R}} \Theta_i^p(\mathbf{z}^{agg}) - \sum_i z_i^{agg} = \sum_p \tau^p - \sum_{i \in \mathcal{R}} z_i^{agg}, \tag{3.44}$$

where the last equality follows from the fact that the dynamics is exact target. Moreover,

$$\sum_{i \in \mathcal{R}} \frac{\partial \dot{z}_i^{agg}}{\partial z_j^{agg}} = \frac{\partial \left(\sum_{i \in \mathcal{R}} \dot{z}_i^{agg} \right)}{\partial z_j^{agg}} = -1. \tag{3.45}$$

Hence, the Jacobian is diagonally dominant by columns. It thus follows that $\mu_1(J(\mathbf{z}^{agg})) = -1$ for every $\mathbf{z}^{agg}$, and exponential convergence of aggregate flows to a globally exponentially stable fixed point follows from Proposition 3.2. The convergence of population route flows follows from Lemma 3.3, and the statement on series of parallel networks follows from Lemma 3.2. □

Proposition 3.3 states that monotone decoupled exact target dynamics that depend on aggregate flows admit a globally asymptotically stable fixed point on series of parallel networks. Since the logit dynamics satisfies all the required properties, the result applies to the logit dynamics. Moreover, the unique fixed point of the logit dynamics approaches the set of the Wardrop equilibria as the noise vanishes, as proved in the next theorem.

Theorem 3.2 ([70]). *Consider a heterogeneous game ($\mathcal{G}$, $\mathcal{P}$, $\mathbf{d}, \boldsymbol{\tau}$), and consider the corresponding logit dynamics defined in* (3.13). *Then:*

1. *If $\mathcal{G}$ is a parallel network, the dynamics* (3.31) *admits a globally asymptotically stable fixed point, and in particular the aggregate flows $\mathbf{z}^{agg}$ converge exponentially fast.*
2. *If $\mathcal{G}$ is a series of parallel networks, then the logit dynamics admits a globally asymptotically stable fixed point $\mathbf{z}_\eta$.*
3. *In the limit of the vanishing noise, the unique fixed point of the dynamics approaches the set of the limit equilibria, i.e., $\mathbf{z}_\eta \xrightarrow{\eta \to +\infty} \overline{\mathcal{Z}^*}$.*

Proof. Points 1 and 2 follow from Proposition 3.3, by noticing that the logit dynamics is monotone, exact target, decoupled, and its interaction kernels depend on the aggregate flows. Point 3 follows from Theorem 3.1. □

The next example illustrates numerical simulations of the logit dynamics on a parallel network, confirming our theoretical results.

Example 3.1. *Consider a parallel network with two nodes* o *and* d *linked by two parallel links. Let*

$$\begin{cases} d_1^1(x) = x, \\ d_2^1(x) = x+1, \\ d_1^2(x) = x, \\ d_2^2 = 2x, \end{cases} \qquad \begin{cases} \tau_1 = 2, \\ \tau_2 = 1. \end{cases}$$

Such an assignment satisfies the hypotheses under which a continuum of equilibria exists (see Example 2.8). The unique equilibrium in terms of aggregate flows is $(f^*)_1^{agg} = 2, (f^*)_2^{agg} = 1$. *Thus the continuum of equilibria is*

$$\begin{cases} f_1^1 = f_1^1, \\ f_2^1 = 2 - f_1^1, \\ f_1^2 = 2 - f_1^1, \\ f_2^2 = f_1^1 - 1, \end{cases}$$

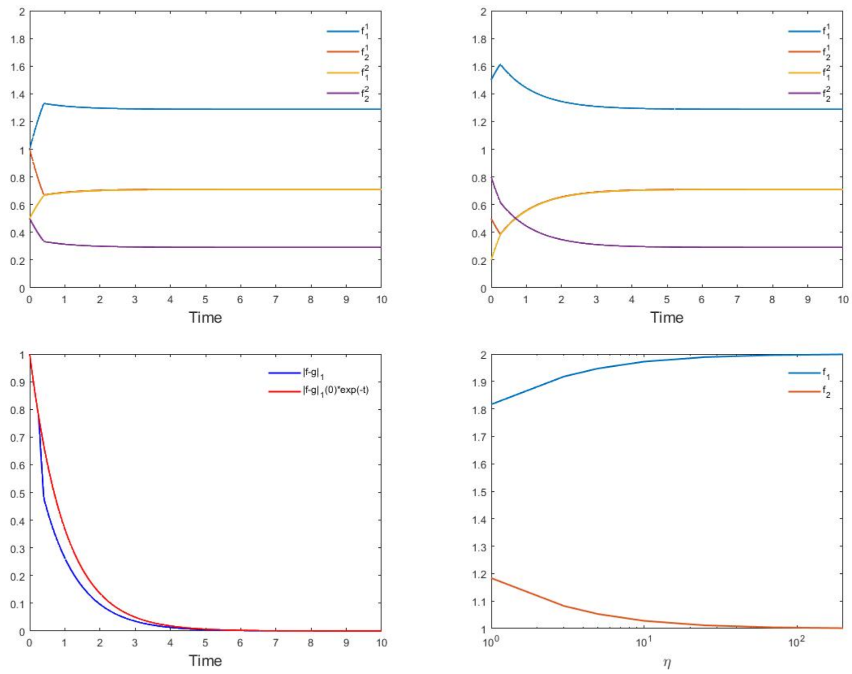


Fig. 3.1 Numerical simulations of the logit dynamics of the game in Example 2.8, with $\eta = 100$. *Above*: two initial conditions converging to same fixed point. *Bottom-left*: l_1 distance between the two trajectories, compared with an exponential behaviour. *Bottom-right*: the unique fixed point of the dynamic expressed in terms of the aggregate flows, as a function of the noise level.

with $f_1^1 \in [1,2]$. *The top plots in Figure 3.1 show two simulations with* $\eta = 100$ *and different initial conditions. For both the initial conditions the dynamics converge to the same point, coherently with Theorem 3.2. Moreover, the bottom left plot shows given the two trajectories, projected in the space of the aggregate flows, converge each other faster than exponentially, confirming the theoretical results. The bottom-right plot instead shows that as* η *increases (thus, the noise decreases) the fixed point, projected in the aggregate flow space, approaches the set of the Wardrop equilibria of the game.*

The stability results established so far require the network to be parallel or a series of parallel networks. In particular, in Theorem 3.2 we show that the logit dynamics admits a globally attractive fixed point if the network is a series of parallel networks, and this fixed point approaches the set of the Wardrop equilibria of the game as the noise vanishes. Motivated by numerical examples, in the next section

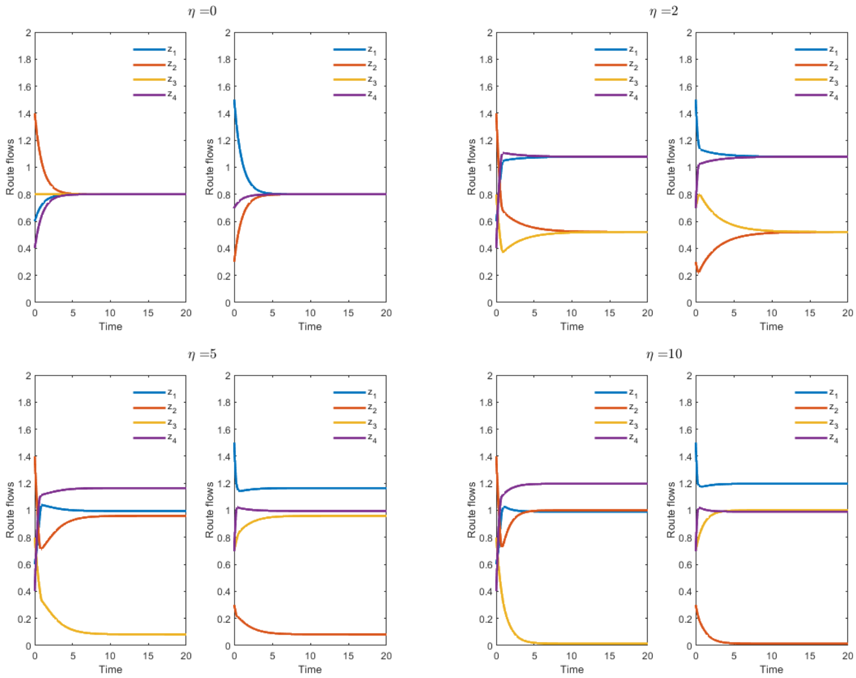


Fig. 3.2 Numerical simulations of the logit dynamics corresponding to the heterogeneous routing game in Example 2.8. We illustrate two trajectories corresponding to different initial conditions for many values of η. The trajectories are projected in the space of the aggregate route flows.

we investigate the behaviour of the dynamics when the network is arbitrary, in both the limits of large and vanishing noise.

3.6 Logit dynamics on arbitrary networks

This section is devoted to analyse the behaviour of the logit dynamics on arbitrary networks. Our main results are the following:

1. in the limit of large noise, the logit dynamics admits a globally exponentially stable fixed point;
2. in the limit of vanishing noise, every strict equilibrium is locally asymptotically stable.

The first result follows from contractive systems theory, in particular from Proposition 3.2. The local stability of strict equilibria relies on linearisation techniques. Before establishing the theoretical results, we provide a motivating example.

Example 3.2. *We provide numerical simulations of the logit dynamics corresponding to the heterogeneous game presented in Example 2.8. The game possesses three essentially different Wardrop equilibria, two of them strict. In Figure 3.2 two trajectories corresponding to different initial conditions are illustrated. The simulations are performed with four different values of* η*, and the trajectories are projected in the space of the aggregate network flows. When* $\eta = 0$ *(infinite noise), both the trajectories converge to a fixed point in which all the populations randomize between the routes. As* η *increases, the asymptotic state of the system varies, but the trajectories still converge to a unique fixed point. For a larger* η*, the system exhibits a bifurcation. Specifically, the two trajectories converge to two different fixed points, that approach the two strict equilibria of the game as* η *increases. We observe from Figure 3.3 that the system exhibits a pitchfork bifurcation. If* η *is smaller than a critical threshold* $\eta^* \simeq 3.22$*, then the system admits a globally asymptotically stable fixed point. As* η *grows, such a fixed point becomes unstable, and two fixed points approaching the strict equilibria arise. The unstable fixed point corresponds to the third Wardrop equilibrium. We thus argue that also the third Wardrop equilibrium is a limit equilibrium, but in contrast with the strict equilibria, numerical simulations reveal its instability. To further verify the instability of the middle equilibrium, we report in Figure 3.4 the eigenvalue with the largest real part of the Jacobian of the system, computed in the unstable fixed point as a function of* η*. As expected, the eigenvalue becomes positive about* η^**.*

Motivated by the numerical example, we investigate the behaviour of the logit dynamics on arbitrary network, and provide two theoretical results. The next theorem characterizes the behaviour of the logit dynamics in the limit of large noise.

Theorem 3.3. *Consider a heterogeneous routing game on an arbitrary network, and consider the corresponding logit*(η) *defined in* (3.13)*. Then, for every* $k \in (0,1]$*, there exist* η_k *such that for* $\eta \in [0, \eta_k]$ *logit*(η) *admits a globally exponentially stable fixed point with rate k.*

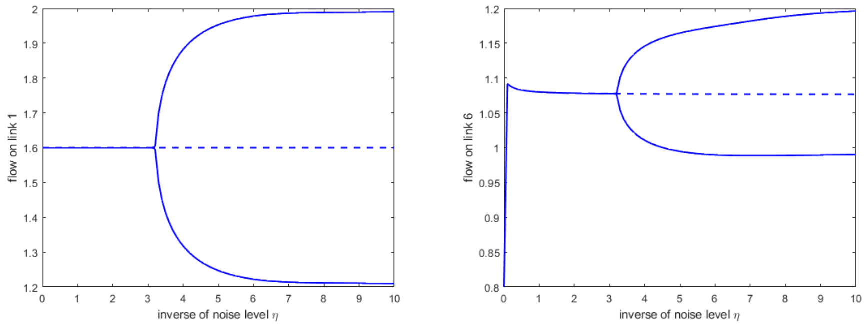


Fig. 3.3 Bifurcation diagram of logit(η) corresponding to the game in Example 2.8. For simplicity we plot only two of the six aggregate network flow components, but similar diagrams may be shown for other components.

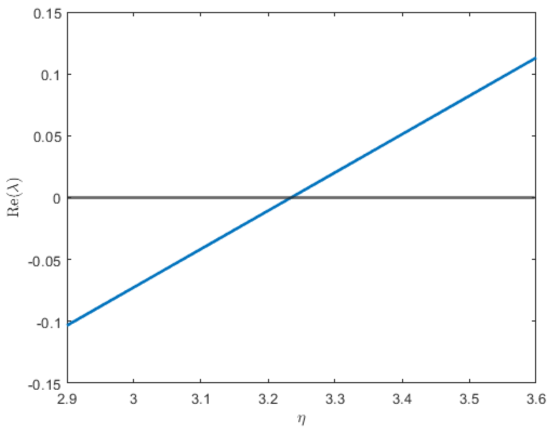


Fig. 3.4 The eigenvalue with largest real part of the Jacobian of the logit dynamics, computed in the fixed point that switches from stable to unstable as η grows. In accordance with the bifurcation diagram in Figure 3.3, the fixed point become unstable around $\eta^* = 3.22$.

Proof. For every $\mathbf{z}$, it holds:

$$\frac{\partial \dot{z}_i^p(\mathbf{z},\eta)}{\partial z_j^q} = \eta\tau^p \frac{\exp\left(-\eta c_i^p(\mathbf{z})\right)\sum_s \exp\left(-\eta c_s^p(\mathbf{z})\right)\left(\frac{\partial c_s^p(\mathbf{z})}{\partial z_j^q} - \frac{\partial c_i^p(\mathbf{z})}{\partial z_j^q}\right)}{\left(\sum_s \exp\left(-\eta c_s^p(\mathbf{z})\right)\right)^2} - \delta_{ij}\delta_{pq}, \tag{3.46}$$

thus the Jacobian reads

$$J(\mathbf{z},\eta) = \eta M(\mathbf{z},\eta) - \mathbf{I}, \tag{3.47}$$

where $M : \mathcal{Z} \times \mathbb{R}_+ \to \mathbb{R}^{\mathrm{RP}\times\mathrm{RP}}$. For every network, $M(\mathbf{z},\eta)$ satisfies two properties:

1. the diagonal is nonpositive, because for every populations p,q and routes j,i,

$$\frac{\partial c_i^p(\mathbf{z})}{\partial z_j^q} \leq \frac{\partial c_i^p(\mathbf{z})}{\partial z_i^p};$$

2. $M(\mathbf{z},0) = \mathbf{0}_{\mathrm{RP}\times\mathrm{RP}}, \quad$ for every $\mathbf{z} \in \mathbb{R}_+^{\mathrm{R}}$.

Thus, for every network and assignment of delay functions, it holds $J(\mathbf{z},0) = -\mathbf{I}$, which implies

$$\mu_1(J(\mathbf{z},0)) = -1 \quad \forall\, \mathbf{z} \in \mathcal{Z}, \tag{3.48}$$

where the definition of the matrix measure μ_1 is given in Definition B.9. Since $M(\mathbf{z},\eta)$ is continuously differentiable in η, it follows that for every $k \in (0,1]$, there exist $\eta_k \geq 0$ such that for every $\mathbf{z}$ and $\eta \in [0,\eta_k]$,

$$\mu_1(J(\mathbf{z},\eta)) \leq -k. \tag{3.49}$$

The existence of a globally exponentially stable fixed point with rate k thus follows from Proposition 3.2. □

Theorem 3.3 characterizes the behaviour of the logit dynamics in the large noise regime. Note that such a result holds on every arbitrary network. In the next theorem, we study the the logit dynamics on arbitrary networks in the vanishing noise regime. In particular, Theorem 3.4 states that strict equilibria are locally asymptotically stable without any assumption on the network topology, where stability of Wardrop equilibria has to be meant as stability of fixed points converging to it.

Theorem 3.4. *Consider a heterogeneous routing game on an arbitrary network, and consider the corresponding logit dynamics defined in* (3.13)*. Then, for every strict equilibrium* $\mathbf{z}^*$*, fixed points of logit*(η) *such that* $\lim_{\eta\to+\infty}\mathbf{z}_\eta = \mathbf{z}^*$*. are locally asymptotically stable under logit*(η) *in the limit* $\eta \to +\infty$*.*

Proof. Consider a strict equilibrium $\mathbf{z}^*$. It follows from Theorem 3.1 that $\mathbf{z}^*$ admits a sequence of fixed points $\mathbf{z}_n \in \Omega_{\eta_n}$ such that $\lim_{n\to+\infty}\mathbf{z}_n = \mathbf{z}^*$. We aim at studying the linear stability of $\mathbf{z}_n$. We recall from (3.46) that for every $\mathbf{z}$, the Jacobian is in the form

$$J(\mathbf{z},\eta) = \eta M(\mathbf{z},\eta) - \mathbf{I}. \tag{3.50}$$

The linear stability of a fixed point $\mathbf{z}_n$ in the vanishing noise regime thus depends on the eigenvalues of the matrix $\lim_{n\to+\infty}\eta_n M(\mathbf{z}_n,\eta_n)$, whose real part has to be less than 1 to ensure stability. Ordering the components by $\{z_1^1,\cdots,z_{\mathrm{R}}^1,\cdots,z_{\mathrm{P}}^1,\cdots,z_{\mathrm{P}}^{\mathrm{R}}\}$, since $\partial\dot{z}_i^p(\mathbf{z},\eta)/\partial z_j^q$ does not depend on q (because user cost functions depend on aggregate flows), M is in the form

$$M = \begin{pmatrix} M_1 & \cdots & M_1 \\ \vdots & \ddots & \vdots \\ M_{\mathrm{P}} & \cdots & M_{\mathrm{P}} \end{pmatrix}. \tag{3.51}$$

Consider a population p that uses only route i, i.e., $c_i^p(\mathbf{z}^*) < c_r^p(\mathbf{z}^*)$ for every route $r \neq i$. Thus, for every route $r \neq i$,

$$\lim_{n\to+\infty}\eta_n \frac{\exp(-\eta_n c_i^p(\mathbf{z}_n))\exp(-\eta_n c_r^p(\mathbf{z}_n))}{(\sum_s \exp(-\eta_n c_s^p(\mathbf{z}_n)))^2} = 0, \tag{3.52}$$

since, as $\eta_n \to +\infty$, the numerator is dominated by the term $\exp(-2\eta c_i)$ of the denominator. This implies that $\lim_{n\to+\infty}\eta_n (M_p)_{ir}(\mathbf{z}_n,\eta_n) = 0$ for every $r \neq i$. By similar arguments, $\lim_{n\to+\infty}\eta_n (M_p)_{jr}(\mathbf{z}_n,\eta_n) = 0$ for every suboptimal route $j \neq i$ and for every r. We now show that $\lim_{n\to+\infty}\eta_n (M_p)_{ii}(\mathbf{z}_n,\eta_n) = 0$ for every i. To this end, note that, under logit(η), for every route i and population p it holds

$$\frac{z_i^p + \dot{z}_i^p}{\tau^p} = \frac{\exp(-\eta c_i^p(\mathbf{z}))}{\sum_{s\in\mathcal{R}}\exp(-\eta c_s^p(\mathbf{s}))}. \tag{3.53}$$

Thus, since fixed points by definition satisfy $\dot{z}_i^p = 0$, then

$$\begin{aligned} M_{ii}^p(\mathbf{z}_n, \eta_n) &= \tau^p \frac{\exp(-\eta c_i^p(\mathbf{z})) \sum_s \exp(-\eta c_s^p(\mathbf{z})) \left(\frac{\partial c_s^p(\mathbf{z})}{\partial z_i^p} - \frac{\partial c_i^p(\mathbf{z})}{\partial z_i^p} \right)}{(\sum_s \exp(-\eta c_s^p(\mathbf{z})))^2} \\ &= \frac{z_i^p}{\tau^p} \sum_{s \in \mathcal{R}} z_s^p \left(\frac{\partial c_s^p(\mathbf{z})}{\partial z_i^p} - \frac{\partial c_i^p(\mathbf{z})}{\partial z_i^p} \right) \end{aligned} \tag{3.54}$$

Observe from (3.13) that $(\mathbf{z}_\eta)^p$ converges to $\tau^p \delta_i$ exponentially fast in η, since the equilibrium is strict. Hence, we conclude that $\lim_{n \to +\infty} \eta_n M_{ii}^p(\mathbf{z}_n, \eta_n) = 0$. Since we have proved that every element of $\lim_{n \to +\infty} \eta_n M^p(\mathbf{z}_n, \eta_n)$ converges to zero, the linear stability follows from form of the Jacobian in (3.46). The local asymptotic stability thus follows from Proposition B.1. □

This result must be compared with the existing literature. It is known that the notion of evolutionary stable state allows to establish results on the local asymptotic stability of fixed points converging to it in the limit of vanishing noise. Specifically, in [17, Theorem 8.4.6] it is proved that, given an arbitrary interior evolutionary stable state $\mathbf{z}$ of a population game, for every neighborhood of $\mathbf{z}$ and large enough η, there exists one and only one fixed point of logit(η). Furthermore, this fixed point is locally asymptotically stable in the limit of vanishing noise. Although strict equilibria are evolutionary stable states, they are not interior, thus violating one of the hypotheses needed for [17, Theorem 8.4.6] and making our result original.

In the next section we discuss conjectures and generalizations of our theoretical results.

3.7 Conjectures and generalizations

3.7.1 Logit dynamics on nearly parallel networks

Theorem 3.2 states that on series of parallel networks the logit dynamics admits a globally asymptotically stable fixed point. However, there is a gap between networks that admit an essentially unique equilibrium (series of nearly parallel (SNP), as established in Section 2.5) and networks that admit a globally asymptotically stable fixed point under the logit dynamics (series of parallel, see Theorem 3.2). Motivated

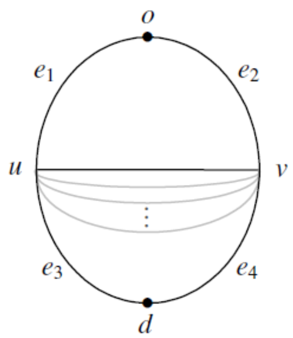
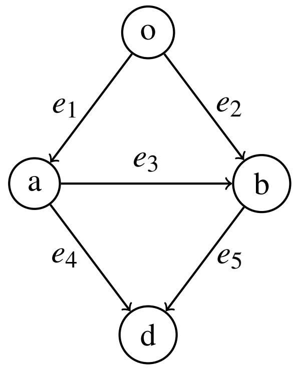


Fig. 3.5 If the logit dynamics on every network of this class of networks admits a globally exponentially stable fixed point, then the logit dynamics admits a globally asymptotically stable fixed point on every series of nearly parallel networks.

by numerical examples, we conjecture that the existence of a globally asymptotically stable fixed point of the logit dynamics continues to hold for SNP class, which is a superset of series of parallel networks. Note that if one is capable to prove that the logit dynamics admits a globally asymptotically stable fixed point on nearly parallel networks, the result can be extended to SNP due to Lemma 3.2. Moreover, Theorem 3.1 allows to conclude that the unique fixed point approaches the set of the Wardrop equilibria in the limit of vanishing noise. Proving this conjecture is still an open problem. We are currently able to prove that if the logit dynamics admits a globally asymptotically stable fixed point on directed instances of the network in Figure 3.5, then the results holds on every nearly parallel network. Unfortunately, the monotonicity arguments used in Theorem 3.2 do not apply to a such a case, because the routes are not parallel, and the conjecture remains thus open. In the next example we provide simulations of the logit dynamics on the Wheatstone network.

Example 3.3. *Consider the heterogeneous game on the Wheatstone network in Figure 3.6. Since the Wheatstone network is nearly parallel, the equilibrium of this game is essentially unique. However, for this assignment of throughput and delay functions the game admits a continuum of equilibria in terms of population flows. In particular, one can verify that equilibrium route flows are in the parametric form*

$$z_1^1 = z_1^1, \quad z_2^1 = 0, \quad z_3^1 = 1 - z_1^1, \quad z_1^2 = 1 - z_1^1, \quad z_2^2 = 2, \quad z_3^2 = z_1^1, \tag{3.55}$$



p	$d_1^p(x)$	$d_2^p(x)$	$d_3^p(x)$	$d_4(x)$	$d_5^p(x)$	τ^p
1	x	20	2x	5x	x	1
2	x	2x	2x	5x	x	3

Fig. 3.6 A heterogeneous game on the Wheatstone network possessing a continuum of equilibria.

with $z_1^1 \in [0,1]$. *Let* $r_1 = (e_1, e_4)$, $r_2 = (e_2, e_5)$, $r_3 = (e_1, e_3, e_5)$. *To verify that* (3.55) *is a Wardrop equilibrium notice that under flows* (3.55) *the route user costs are*

$$c_1^1 = c_3^1 = 7 < 20 = c_2^1, \quad c_1^2 = c_2^2 = c_3^2 = 7. \tag{3.56}$$

Notice also that for this particular network there is a one-to-one correspondence between network flows and route flows. The equilibrium flows (3.55) *induce a unique aggregate route flow (and thus a unique aggregate network flow)*

$$z_1^{agg} = 1, \quad z_2^{agg} = 2, \quad z_3^{agg} = 1, \tag{3.57}$$

in accordance with Proposition 2.5. In Figure 3.7 it is shown that, although the game possesses a continuum of Wardrop equilibria, the logit dynamics converges to a unique asymptotically stable fixed point for two different initial conditions. We used for the simulations $\eta = 20$. *Such a value proves to be large enough to verify that the fixed point of the dynamics approaches the set of the Wardrop equilibria, in accordance with Theorem 3.1.*

3.7.2 Limit equilibria of heterogeneous routing games

Theorem 3.1 states that the set of the fixed points of the logit dynamics in heterogeneous routing games approaches the set of the equilibria of the game as the noise vanishes. However, not every Wardrop equilibrium of the game is the limit of fixed points of the dynamics, i.e., $\overline{\mathcal{Z}^*} \neq \mathcal{Z}^*$. We show in Theorem 3.1 that every strict equilibrium is a limit equilibrium (and also show in Theorem 3.4 that fixed points

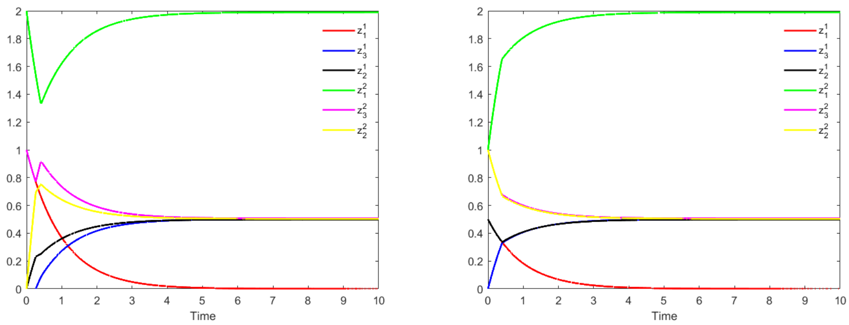


Fig. 3.7 Numerical simulations of the logit dynamics with $\eta = 20$ corresponding to the game in Figure 3.6. The two plots show two trajectories with different initial conditions, converging to the same fixed point.

converging to strict equilibria are locally asymptotically stable), but a complete characterization of the set of the limit equilibria $\overline{\mathcal{Z}^*}$ is not provided. Our conjecture is that every connected component of Wardrop equilibria admits one and only one limit equilibrium. The conjecture is motivated by numerical examples. Indeed, in Example 3.2 the game admits three isolated equilibria, all of them being limit equilibria (although one of them unstable for small noise $1/\eta$). The games in Examples 3.1 and 3.3 admit connected sets of Wardrop equilibria, but the logit dynamics seems to select one of the infinite equilibria when the noise tends to vanish.

3.7.3 Local stability of quasistrict equilibria

Theorem 3.1 states that every strict equilibrium is a limit equilibrium, and Theorem 3.4 proves that fixed points approaching it are locally asymptotically stable. We conjecture that the local stability results can be extended to the case of quasi-strict equilibria that are strict for $\mathrm{P}-1$ populations, and quasi-strict for the remaining population, with the remaining population using only two routes. This conjecture would be proved if we were able to prove that equilibria in this form are limit equilibria of the game. From there on, the stability of fixed points can be proved by using similar techniques as in Theorem 3.4. Also, note that, if one additionally assumes that the delay functions of the game are strictly increasing, then quasi-strict equilibria in this form are isolated. Hence, the prove of this conjecture would automatically follow from the conjecture presented in the previous section.

3.8 Conclusion

In this chapter we investigate the asymptotic behaviour of evolutionary dynamics in routing games, focusing specifically on the logit dynamics. While the behaviour of the logit dynamics in homogeneous routing games is well-known in the literature, a theory for heterogeneous routing games is still missing in the literature. Specifically, it is known that the logit dynamics in homogeneous routing games admits a globally asymptotically stable fixed point that approaches the set of the Wardrop equilibria of the game in the limit of vanishing noise. We extend this result to heterogeneous routing games under the assumption that the network is parallel or it is the series composition of parallel networks, and additionally prove that the existence of an asymptotically stable fixed point holds even for a more general class of dynamics. We then analyse the logit dynamics in heterogeneous routing games on arbitrary networks. We show that in the limit of vanishing noise the fixed points of the dynamics converge to Wardrop equilibria of the game, and additionally show that strict equilibria are locally asymptotically stable. Additionally, we show that in the limit of large noise the logit dynamics admits a globally exponentially stable fixed point on arbitrary networks.

For the future, we aim at proving the conjectures presented in Section 3.7. Specifically, we aim at extending to series of nearly parallel networks the global stability results established for series of parallel networks. Furthermore, we aim at characterizing the set of the limit equilibria that are approximated by fixed points of the logit dynamics, and find sufficient and necessary conditions under which a limit equilibrium is locally asymptotically stable. Other research lines include the generalizations of the stability results to different evolutionary dynamics and to more complicated routing games, e.g., to consider multiple origin-destination pairs.

Chapter 4

Network design of transportation networks

4.1 Introduction

Due to increasing populations living in urban areas, many cities are facing the problem of traffic congestion, which leads to increasing levels of pollution and massive waste of time and money [2]. The problem of mitigating congestion has been tackled in the literature from two main perspectives. One approach is to indirectly influence the behaviour of the drivers by incentive-design mechanisms, with the goal of minimizing the inefficiencies due to the autonomous uncoordinated decisions of users. A second approach is to intervene on the transportation network directly, by building new roads or enlarging existing ones. The corresponding *network design problem* (i.e., the problem of optimizing the intervention on a transportation network subject to some budget constraints, see e.g. [10]) is very challenging because of its bi-level nature [11], i.e., it involves a network intervention optimization problem given the flow distribution for that particular network. For simplicity, we work in the setting of homogeneous routing games, i.e., we assume that each link of the network is endowed with a delay function that is common to all the users, and the flow distributes according to a Wardrop equilibrium, taking routes with minimum delay. A characterization of Wardrop equilibrium is used to construct the lower level of the bilevel network design problem. We assume that the goal of the planner is to minimize the total travel time at the equilibrium, i.e., the social cost of the game according to Definition 2.16. Since the considered routing game is homogeneous, the theoretical results contained in the previous chapter guarantee that stability of equilibrium flows under evolutionary dynamics. An additional reason to adopt the homogeneous setting is that the notion of social cost in heterogeneous games is not even well defined, since the delay functions on the links are not unique and vary among the populations. Indeed, if the different user cost functions are due to different information on the state of the streets, then the social cost should be defined in terms of *real* delay functions that do not depend on the population; if instead delay functions correspond to a actual different user cost over the link set (for instance due to different trade-offs between travel time and fuel consumption) it seems more suitable to measure the social cost as the sum of the user cost incurred by every user.

In this work we study a special class of network design problem (NDP), where the planner can improve the delay function of a single link. For this class of NDPs, our first main result provides an analytical characterization of the social cost variation corresponding to an intervention on a particular link under a regularity assumption, which states that the links that carry positive flow remain unchanged with an intervention. This assumption, which is not new in the traffic equilibrium literature (see e.g. [83, 84]) leads to a characterization of Wardrop equilibria using a system of linear equations and enables representing link interventions as rank-1 perturbations of the system. We show that this assumption is satisfied provided that the total incoming flow to the network is large enough and the network is series-parallel, which may be of independent interest. We exploit the structure of our characterization and linearity of delay functions to express the social cost variation using the effective resistance of a link (i.e., between the end-points of the link), defined with respect to a related resistor network. Computing the effective resistance of a single link requires the solution of a linear system with a matrix whose size scales with the size of the network (we indistinctly refer to the size of the network as the cardinality of the node and the link sets, implicitly assuming that transportation networks are sparse in a such a way that the average degree of the nodes is independent of the number of nodes, inducing then a proportionality between the number of nodes and links). Hence, solving the NDP requires the solution of E of these problems, with E denoting the number of links. Since this can be computationally intractable for large networks, our second main result proposes a method based on Rayleigh's monotonicity laws to approximate the effective resistance of each link with a number of iterations independent of the network size, thus leading to a significant reduction of complexity.

The key idea is that the effective resistance between two adjacent nodes i and j depends mainly on the local structure of the network around the two nodes (i.e., the set of nodes $\mathcal{N}_d(\{i,j\})$ that are at distance no greater than a small given constant d from at least one of i and j), and may therefore be approximated by performing only local computations. Since for networks with bounded degrees (as typical in traffic networks, think for instance of the bidimensional square grid) the size of $\mathcal{N}_d(\{i,j\})$ does not scale with the network size, we can guarantee that the approximation error and computational complexity of our method also do not scale. Then, we provide sufficient conditions on the network under which the approximation error vanishes asymptotically in the limit of infinite networks, and show simulations on synthetic and real networks. Afterwards, we use similar arguments to evaluate the effect of adding a link instead of modifying a pre-existing one. However, since the end-nodes of the new link might in principle be separated by a large distance in the original network, the approximation methods do not apply to the case.

In our work we consider a special case of a NDP. These problems have been formalized in the last decades via many different formulations, as discussed in Chapter 1. We stress that most of the literature focuses on finding time polynomial algorithms to approximately solve NDPs in their most general form. As noted above, we instead consider a problem that can be solved with a polynomial algorithm by simply enumerating all the candidate links and computing the social cost corresponding to the intervention on each of those links. Our main contribution is to define a simplified, more intuitive and tractable approach to solve such a design problem in quasilinear time instead of polynomial, as well as providing intuition and a complete new formulation. For the future we aim at extending our techniques to more general cases, like the multiple interventions case. Unfortunately, as shown in the conclusive section, our objective function is not submodular and thus we do not have any guarantees on the performance of greedy algorithms. Since we assume that delay functions are affine, our NDP formulation is strictly related to marginal toll literature. Indeed, assuming that the intervention improves the slope of a link e from a_e to $\tilde{a}_e$, leading to $\tilde{d}_e(f_e) = d_e(f_e) - (a_e - \tilde{a}_e)f_e$, our intervention is equivalent to adding a scaled negative marginal toll on a street, where marginal tolls are defined in the functional form $f_e(d_e'(f_e)) \propto f_e$. In the literature, the problem of optimal tolling has been widely explored, also dealing with the problem of the support of the Wardrop equilibrium varying with the intervention, i.e., without imposing restrictive assumptions. However, as NDPs literature, most of the literature on optimal tolling aims at finding conditions under which a general NP-hard problem may be solved in polynomial time. Moreover, to relax our assumption it is often assumed that the network has parallel links, which is unrealistic for transportation networks (see, e.g., [85, 86], where the authors find conditions under which an optimal tolling satisfying certain constraints may be found in polynomial time in networks with parallel links). Another mathematically equivalent setting to ours is to consider users that minimize a combination of the delay function and marginal social cost the associated to their choice, which is in the form $(f_e d_e(f_e))'$, and thus linear in f_e as our intervention. Models like this have been considered for instance in [87], but with different purposes not related to ours. Our work is related to [83, 84], where the authors investigate the sign of total travel time variation when a new path is added to a two-terminal network, under similar assumptions to ours, providing sufficient conditions under which Braess paradox arises. In our work we instead suggest an efficient algorithm to select the best link to improve. As mentioned, the key steps of our approach is to reformulate the NDP in terms of a resistance problem, and also exploit the parallelism between random walks and resistance networks. From a methodological perspective it is worthwhile mentioning that the relation between Wardrop equilibria and electric flows has been first investigated in [88], while the parallelism between random walks and Wardrop equilibria has been explored in [89], although with different purposes. The equivalence between random walks and electrical flows is quite standard and illustrated in Appendix A.1.

To summarize, the contribution of this chapter is two-fold. From a methodological perspective, we provide a method to locally upper and lower bound the effective resistance between adjacent nodes, which may be of a separate interest beyond traffic applications. From the network design perspective, we provide a new formulation of the design problem in terms of resistor networks, and we exploit our methodological result to approximate in an efficient manner a simplified version of the design problem where a single link can be improved. The chapter is based on [71].

The remainder of the chapter is organized as follows. In Section 4.2 we define the model and formulate the problem as a bi-level programming. In Section 4.3 we rephrase it in terms of resistor networks, and discuss our main assumption. In Section 4.4 we provide our method to approximate effective resistance between neighboring nodes and our algorithm to select the optimal link for the intervention. In Section 4.5 we analyse the asymptotic behaviour of the bounds in the limit of infinite networks.

In Section 4.6 we show some simulations over relevant networks. In Section 4.7 we use similar techniques to extend our analysis to the case of adding a link instead of modifying a pre-existing one, and discuss the case of multiple interventions. Finally, in the conclusive section, we summarize the work and discuss future research lines.

4.2 Model and problem formulation

We consider a homogeneous routing game as defined in Section 2.5, with the additional assumption that the delay functions are assumed affine, and strictly increasing, i.e.,

$$d_l(f_l) = a_l f_l + b_l, \quad a_l > 0, \quad b_l \geq 0, \quad \forall l \in \mathcal{E}.$$

We let $A \in \mathbb{R}^{\mathcal{E}\times\mathcal{E}}$ and $\mathbf{b} \in \mathbb{R}^{\mathcal{E}}$ be

$$A := \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{\mathrm{E}} \end{bmatrix}, \quad \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{\mathrm{E}} \end{bmatrix}.$$

Definition 4.1 (Affine routing game)**.** *An* affine routing game *is a quadruple* $(\mathcal{G}, A, \mathbf{b}, \tau)$*, where* $\mathcal{G}$ *is a two-terminal directed multigraph.*

We assume that every link belongs to at least a route, otherwise it can be removed without loss of generality. We consider the problem of a planner that aims at minimizing the social cost by improving a link of the network. We propose as intervention to rescale the slope of one link l by a scaling parameter $\kappa > 1$, so that the slope of the link l gets reduced from a_l to $\tilde{a}_l = a_l/\kappa$. This intervention may correspond for instance to adding a new lane in a street. Actually, every intervention on a single link may be seen as a rank-1 perturbation of the system and may be handled by our method (see Section 4.3.1 for more details). We aim at identifying which link should be selected by the planner to minimize the social cost. Let $\mathbf{f}^*(l)$ and $C(\mathbf{f}^*(l))$ denote the Wardrop equilibrium when the slope of the link l is rescaled, and the corresponding social cost, respectively. Hence, the problem can be expressed as follows.

Problem 4.1. *Let $(\mathcal{G}, A, \mathbf{b}, \tau)$ be an affine routing game and $\kappa > 1$ be the scaling parameter. Find link l^* such that*

$$l^* \in \arg\min_{l \in \mathcal{E}} \quad C(\mathbf{f}^*(l)).$$

We stress the fact that Problem 4.1 is bi-level, in the sense that the planner optimizes the intervention over the link set, but the cost function is a function of the Wardrop equilibrium $\mathbf{f}^*$, which in turn is the solution of the optimization problem (2.15). Problem 4.1 can be solved by a brute force approach, by enumerating all the links and computing the corresponding equilibrium $\mathbf{f}^*(l)$ by solving the convex program (2.15) with $d_l(f_l) = \tilde{a}_l f_l + b_l$ instead of the original $d_l(f_l)$. In this work we propose a method that, given $\mathbf{f}^*$ before the intervention (which is assumed to be observable and therefore known) and other electrical quantities computed on a resistance network related to the original unperturbed transportation network, provides an upper and lower bound to $C(\mathbf{f}^*(l))$ with a computational complexity that does not scale with the size of the network. The main idea is that the effect of perturbing a link may be well approximated by looking at a local portion of the network. Our method works under the assumption that the network is sparse in such a way that the average degree of the nodes does not depend on the size of the network, and under the assumption that the set of the used links does not change after the intervention. The first assumption is suitable for traffic networks, and the second one is not new in the literature on intervention in traffic networks [83, 84]. We provide a more detailed discussion on this assumption in Section 4.3.3.

Remark 4.1. *Observe that, although negative marginal tolls and our interventions modify the delay functions in identical way, the resulting optimization problem is different. In fact, the performance of interventions in optimal tolling literature are measured by using the old delay functions under the new Wardrop equilibrium without taking into account the toll explicitly in the cost, i.e., with delay functions $d_l(x) = a_l x + b_l$. Instead, in our optimization problem both the functional form of the social cost and the Wardrop equilibrium are modified with an intervention, i.e., the social cost is computed using for the link l the delay function $d_l(x) = \tilde{a}_l x + b_l$.*

4.3 An electrical formulation

In this section we provide an electrical formulation for Problem 4.1 in terms of resistor networks. We do this in two steps. First, we exploit the fact that, under the assumption that the support of the Wardrop equilibrium does not change due to the intervention, modifying the slope of one link is equivalent to introducing a rank-1 perturbation in the KKT conditions of (2.15). Then, we relate this KKT formulation to electrical quantities. Furthermore, we discuss the assumption on the support of Wardrop equilibrium not varying after the intervention, proving that it is guaranteed to hold on series-parallel networks provided that the throughput is sufficiently large.

4.3.1 KKT formulation

Since the optimal intervention is studied in the homogeneous setting, and since the delay functions are assumed strictly increasing, the Wardrop equilibrium of the game is unique, and the unique Wardrop equilibrium may be found by solving an optimization problem (see Theorem 2.1). Moreover, as proved in Proposition 2.2, the social cost associated with the unique Wardrop equilibrium may reformulated in terms of the optimal dual solution, obtained by solving the KKT conditions (2.16). For the particular case of affine games the KKT conditions read

$$\begin{cases} a_l f_l^* + b_l + \gamma^*_{\theta(l)} - \gamma^*_{\xi(l)} - \lambda_l^* = 0 & \forall l \in \mathcal{E}, \\ \sum_{l \in \mathcal{E}:\theta(l)=i} f_l - \sum_{l \in \mathcal{E}:\xi(l)=i} f_l + \nu_i = 0 & \forall i \in \mathcal{N}, \\ \lambda_l^* f_l^* = 0 & \forall l \in \mathcal{E}, \\ \lambda_l^* \geq 0 & \forall l \in \mathcal{E}, \\ f_l^* \geq 0 & \forall l \in \mathcal{E}. \end{cases} \tag{4.1}$$

The complementary slackness (third condition) implies that all the links such that $\lambda_e^* > 0$ are not used at the equilibrium, i.e. $f_e^* = 0$. Let $\mathcal{E}_+$ denote the set of such links. Thus, the links in $\mathcal{E}_+$ and the last three conditions of (4.1) can be removed, without affecting the solution of (4.1). With a slight abuse of notation, from now on let $\mathcal{E}$ denote $\mathcal{E} \setminus \mathcal{E}_+$. Thus, the KKT conditions become:

$$\begin{cases} a_l f_l^* + b_l + \gamma_{\theta(l)}^* - \gamma_{\xi(l)}^* - \lambda_l^* = 0 & \forall l \in \mathcal{E}, \\ \sum_{l \in \mathcal{E}:\theta(l)=i} f_l - \sum_{l \in \mathcal{E}:\xi(l)=i} f_l + \nu_i = 0 & \forall i \in \mathcal{N}, \end{cases} \tag{4.2}$$

where the constraint $f_l^* \geq 0$ can now be removed since the solution of (4.2) gives $f_l^* \geq 0$ for every link $l \notin \mathcal{E}_+$. Without loss of generality, we order the nodes in such a way that the origin o and the destination d are the first and the last node respectively. Since for every link l the optimal flow f_l^* depend on $\boldsymbol{\gamma}^*$ only via the difference $\gamma_{\xi(l)}^* - \gamma_{\theta(l)}^*$ due to the fact that B is not full-rank, we set $\gamma_{\mathrm{d}}^* = 0$, which is equivalent to removing the last row of B. We thus define $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x} := \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\gamma}_- \end{bmatrix}, \quad \mathbf{y} := -\begin{bmatrix} \mathbf{b} \\ \boldsymbol{\nu}_- \end{bmatrix},$$

where $\boldsymbol{\gamma}_-$ and $\boldsymbol{\nu}_-$ denote respectively $\boldsymbol{\gamma}$ and $\boldsymbol{\nu}$ where the last element of both vectors is removed. Also, $B_- \in \mathbb{R}^{(\mathcal{N} \setminus \mathrm{d}) \times \mathcal{E}}$ denotes the node-link incidence matrix where the last row is removed. Finally, we define $H \in \mathbb{R}^{(\mathcal{N} \setminus \mathrm{d} \cup \mathcal{E}) \times (\mathcal{N} \setminus \mathrm{d} \cup \mathcal{E})}$ as

$$H := \begin{bmatrix} A & -(B_-)^T \\ -B_- & \mathbf{0}_{(\mathrm{N}-1)\times(\mathrm{N}-1)} \end{bmatrix}.$$

With this notation and assuming $\gamma_{\mathrm{d}}^* = 0$, the KKT conditions (4.2) become:

$$H\mathbf{x}^* = \mathbf{y}. \tag{4.3}$$

Because we assumed $\gamma_{\mathrm{d}}^* = 0$, $\mathbf{x}^*$ is unique and the following holds:

$$\mathbf{x}^* = H^{-1}\mathbf{y} = \begin{bmatrix} A^{-1} - KQ^{-1}K^T & -KQ^{-1} \\ -Q^{-1}K^T & -Q^{-1} \end{bmatrix} \mathbf{y}, \tag{4.4}$$

where $K := A^{-1}B_-^T \in \mathbb{R}^{\mathcal{E} \times (\mathcal{N} \setminus \mathrm{d})}$ and $Q := B_- A^{-1} B_-^T \in \mathbb{R}^{(\mathcal{N} \setminus \mathrm{d}) \times (\mathcal{N} \setminus \mathrm{d})}$. As shown in [90], the invertibility of H follows from the invertibility of Q, which we prove in the proof of Theorem 4.1, and the invertibility of A (the delays are strictly increasing). Let $A(l)$, $H(l)$, $Q(l)$ and $K(l)$ denote the matrix A, H, Q and K after the intervention on link l.

Proposition 4.1. *Let $l \in \mathcal{E}$, and consider the modified game $(\mathcal{G}, A(l), \mathbf{b}, \tau)$ obtained by changing the slope of link l from a_l to $\tilde{a}_l = a_l/\kappa$ and construct the corresponding primal and dual solution $\mathbf{x}^*(l)$ as in* (4.4). *Then,*

$$C(\mathbf{f}^*) - C(\mathbf{f}^*(l)) = \tau(\gamma_\mathrm{o}^* - \gamma_\mathrm{o}^*(l)),$$

where γ_o^* and $\gamma_\mathrm{o}^*(l)$ are the $(\mathrm{E}+1)-th$ component of $\mathbf{x}^*$ and $\mathbf{x}^*(l)$ respectively.

Proof. The statement follows from Proposition 2.2, and from $\gamma_\mathrm{d}^* = 0$. □

Since τ is a given constant of the problem, Proposition 4.1 states that the goal of the planner should be to select the link l minimizing $\gamma_\mathrm{o}^*(l)$, that is, the optimal lagrangian multiplier of the origin after the intervention on the link l. Note from (4.4) that

$$\boldsymbol{\gamma}^* = Q^{-1}(K^T\mathbf{b} + \boldsymbol{\nu}_-). \tag{4.5}$$

A brute force method would require the computation of $\gamma_\mathrm{o}^*(l)$ for every candidate link l by substituting in (4.5) the corresponding quantities evaluated after the intervention. A natural question is whether it is possible to evaluate $\gamma_\mathrm{o}^*(l)$ for every link l without recomputing explicitly the first component of $\boldsymbol{\gamma}^*(l)$.We shall see in Section 4.3.2 that under the following assumption the answer is positive, specifically the social cost variation may be rephrased in terms of electrical quantities computed on a related resistor network.

Assumption 4.1. *Let $\mathcal{E}_+(l)$ be the set of links e for which $\lambda_e^*(l) > 0$ in the Wardrop equilibrium of $(\mathcal{G}, A(l), \mathbf{b}, \tau)$. We assume that $\mathcal{E}_+(l) = \mathcal{E}_+$ for all $l \in \mathcal{E}$.*

The intuition is that under Assumption 4.1 the KKT conditions (4.3) before and after the intervention on the link l involve the same set of links and differ in the value of a_l only, allowing therefore to handle the intervention as rank-1 perturbation of H, or equivalently (see (4.5)) of Q. In the next section we provide an electrical formulation of Problem 4.1.

4.3.2 Electrical formulation

We start providing an interpretation to Q. The notions used in this section are contained in Appendix A. From the definitions of B_- and A, it follows that for a link

l with $\xi(l) = i, \theta(l) = j$,

$$K_{l:} = \frac{\delta_i^T - \delta_j^T}{a_l}, \tag{4.6}$$

with the convention that $\delta_{\mathrm{d}} = \mathbf{0}_{\mathrm{N}-1}$ (since we removed the destination), and

$$Q_{ij} = \begin{cases} -\sum_{\substack{l \in \mathcal{E}: \\ \xi(l)=i, \theta(l)=j, \text{ or} \\ \xi(l)=j, \theta(l)=i}} \frac{1}{a_l} & \text{if } i \neq j \\ \sum_{\substack{l \in \mathcal{E}: \\ \xi(l)=i \text{ or} \theta(l)=i}} \frac{1}{a_l} & \text{if } i = j. \end{cases} \quad \forall i, j \in \mathcal{N} \setminus \mathrm{d},$$

We remark that Q_{ii} includes also links connecting i with the destination. Note that Q is symmetric by construction. Moreover, Q allows for an interpretation in terms of electrical quantities. We construct the resistor network $\mathcal{G}_R = (\mathcal{N}, \mathcal{L}, W)$ (in the sense of Definition A.5) in the following way. For every link $l \in \mathcal{E}$, we let $\mathcal{L}$ contain $\{\xi(l), \theta(l)\}$, and let $W \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ be

$$W_{ij} := \begin{cases} \sum_{\substack{l \in \mathcal{E}: \\ \xi(l)=i, \theta(l)=j, \text{ or} \\ \xi(l)=j, \theta(l)=i}} \frac{1}{a_l} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Observe that W includes also the destination, and is symmetric by construction. The coefficients a_l correspond to resistances. We let $D \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ denote the diagonal matrix of degree distribution of $\mathcal{G}_R$, i.e., $D = \mathrm{diag}(W\mathbf{1})$, and $P = D^{-1}W \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ its normalized adjacency matrix. Note that the matrix Q may be related to the truncated Laplacian of the resistor network $\mathcal{G}_R$. In particular, by letting $\tilde{D}$ and $\tilde{W}$ the restriction of D and W on $\mathcal{N} \setminus \mathrm{d}$,

$$Q = \tilde{D} - \tilde{W}. \tag{4.7}$$

This is the key point to prove the next theorem. Many notions used for this proof are contained in Section A.2.

Theorem 4.1. *Let $(\mathcal{G}, A, \mathbf{b}, \tau)$ be a routing game, $\kappa > 1$ be the scaling parameter, and suppose Assumption 4.1 holds. The social cost variation corresponding to the intervention on link l, by letting $\xi(l) = i, \theta(l) = j$, is*

$$\Delta C(l) := C(\mathbf{f}^*) - C(\mathbf{f}^*(l)) = \iota \frac{f_l^*(u_i - u_j)}{\frac{1}{\kappa - 1} + \frac{r_{ij}}{a_l}}, \tag{4.8}$$

where ι is a constant independent of l, and r_{ij} is the effective resistance between nodes i and j in the related resistor network $\mathcal{G}_R$, and $\mathbf{u} \in \mathbb{R}^{\mathcal{N}}$ denotes the potential over the nodes of $\mathcal{G}_R$ when the boundary conditions $u_{\mathrm{o}} = 1$ and $u_{\mathrm{d}} = 0$ are imposed.

Proof. Note that an intervention on link l corresponds to a rank-1 perturbation of Q. In particular,

$$Q(l) = Q + \frac{\kappa - 1}{a_l} B_-^l (B_-^l)^T, \tag{4.9}$$

where B_-^l denotes the $l-$th column of B_-. By letting $\xi(l) = i, \theta(l) = j$, we get $B_-^l = \delta_i - \delta_j$. Thus, by Sherman-Morrison formula,

$$Q(l)^{-1} = Q^{-1} - \frac{Q^{-1} B_-^l (B_-^l)^T Q^{-1}}{\frac{a_l}{\kappa - 1} + (B_-^l)^T Q^{-1} B_-^l}. \tag{4.10}$$

Also, for a link l with $\xi(l) = i, \theta(l) = j$, it holds:

$$K(l) = K + \frac{\kappa - 1}{a_l} \delta_l (B_-^l)^T = K + \frac{\kappa - 1}{a_l} \delta_l (\delta_i - \delta_j)^T. \tag{4.11}$$

From $\mathbf{x}^* = H^{-1}\mathbf{y}$, it follows

$$\begin{aligned} \boldsymbol{\gamma}^* - \boldsymbol{\gamma}^*(l) &= Q^{-1}(K^T \mathbf{b} + \boldsymbol{\nu}_-) - \left(Q^{-1} - \frac{Q^{-1} B_-^l (B_-^l)^T Q^{-1}}{\frac{a_l}{\kappa-1} + (B_-^l)^T Q^{-1} B_-^l} \right) \left(K^T \mathbf{b} + \frac{\kappa-1}{a_l} B_-^l \delta_l^T \mathbf{b} + \boldsymbol{\nu}_- \right) \\ &= -\frac{\kappa-1}{a_l} Q^{-1} B_-^l \delta_l^T \mathbf{b} + \frac{Q^{-1} B_-^l (B_-^l)^T Q^{-1}}{\frac{a_l}{\kappa-1} + (B_-^l)^T Q^{-1} B_-^l} \left(K^T \mathbf{b} + \frac{\kappa-1}{a_l} B_-^l \delta_l^T \mathbf{b} + \boldsymbol{\nu}_- \right) \end{aligned} \tag{4.12}$$

We now give an electrical interpretation to the terms in equation (4.12). To this end, we let ${}_{\mathrm{d}}P = \tilde{D}^{-1}\tilde{W} \in \mathbb{R}_+^{(\mathcal{N} \setminus \mathrm{d}) \times (\mathcal{N} \setminus \mathrm{d})}$ denote the transition matrix of the random walk induced by the resistor network killed d, defined as in Section A.3. We recall that ${}_{\mathrm{d}}P$ is substochastic, since the rows referring to nodes pointing to the destination sum to less than one. Furthermore, let ${}_{\mathrm{d}}G = \sum_{t=0}^{\infty} ({}_{\mathrm{d}}P)^t \in \mathbb{R}_+^{(\mathcal{N} \setminus \mathrm{d}) \times (\mathcal{N} \setminus \mathrm{d})}$ denote the Green's function of the killed random walk as defined in Definition A.14. We now prove that Q is invertible. Indeed,

$$Q^{-1} = (\tilde{D} - \tilde{W})^{-1} = (\tilde{D}(\mathbf{I} - {}_{\mathrm{d}}P))^{-1} = (\mathbf{I} - {}_{\mathrm{d}}P)^{-1} \tilde{D}^{-1} = \sum_{t=0}^{\infty} ({}_{\mathrm{d}}P)^t \tilde{D}^{-1} = {}_{\mathrm{d}}G \tilde{D}^{-1}, \tag{4.13}$$

where the first equivalence follows from (4.7) and the penultimate one follows from connectedness of $\mathcal{G}_R$ and (A.17). We now construct $\hat{Q}^{-1} \in \mathbb{R}^{\mathcal{N} \times \mathrm{N}}$ and ${}_{\mathrm{d}}\hat{G} \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$

by adding a zero column and a zero row to Q^{-1}, and ${}_{\mathrm{d}}G$ and $\hat{K} \in \mathbb{R}^{\mathcal{E}\times\mathcal{N}}$ by adding a zero column to K corresponding to the destination. It follows from (4.13) that $\hat{Q}^{-1} = {}_{\mathrm{d}}\hat{G}D^{-1}$, and

$$\begin{aligned}(B^l_-)^T Q^{-1} B^l_- &= (B^l)^T \hat{Q}^{-1} B^l \\ &= \left(\delta_i - \delta_j\right)^T {}_{\mathrm{d}}\hat{G}\tilde{D}^{-1}(\delta_i - \delta_j) \\ &= \frac{{}_{\mathrm{d}}\hat{G}_{ii} - {}_{\mathrm{d}}\hat{G}_{ji}}{D_{ii}} + \frac{{}_{\mathrm{d}}\hat{G}_{jj} - {}_{\mathrm{d}}\hat{G}_{ij}}{D_{jj}} = r_{ij},\end{aligned} \tag{4.14}$$

where the last equivalence follows from (A.23). By letting $\bar{\mathbf{u}}$ denote a scaled version of the potential $\mathbf{u}$, and $\bar{\mathbf{u}}_-$ denote its restriction over $\mathcal{N} \setminus \mathrm{d}$, one can prove that $\bar{\mathbf{u}}_- = Q^{-1}\delta_{\mathrm{o}}$. Indeed, $Q\bar{\mathbf{u}}_- = \delta_{\mathrm{o}}$ implies by (4.7) that for any node $i \neq \mathrm{o}, \mathrm{d}$,

$$D_{ii}(\bar{v}_-)_i - \sum_{j\in\mathcal{N}\setminus\mathrm{d}} W_{ij}(\bar{u}_-)_j = 0, \implies \bar{u}_i = (\bar{u}_-)_i = \sum_{j\in\mathcal{N}\setminus\mathrm{d}} P_{ij}(\bar{u}_-)_j = \sum_{j\in\mathcal{N}} P_{ij}\bar{u}_j, \tag{4.15}$$

where the last equivalence follows from $\bar{u}_{\mathrm{d}} = 0$. Thus, $\bar{\mathbf{u}}$ and $\mathbf{u}$ are both harmonic with respect to P (the definition of harmonic function is established in Definition A.4), both vanishing in the destination but with different condition on the origin, which implies by Proposition A.1 that $\bar{\mathbf{u}}$ is equivalent to $\mathbf{u}$ apart from a multiplicative factor. Plugging this equivalence and (4.14) in (4.12), we get

$$\begin{aligned}\gamma^*_{\mathrm{o}} - \gamma^*_{\mathrm{o}}(l) &= -\tfrac{\kappa-1}{a_l}\delta_{\mathrm{o}}^T Q^{-1} B^l_- \delta_l^T \mathbf{b} + \frac{\delta_{\mathrm{o}}^T Q^{-1} B^l_- (B^l_-)^T Q^{-1}}{\frac{a_l}{\kappa-1} + (B^l_-)^T Q^{-1} B^l_-}\left(K^T\mathbf{b} + \tfrac{\kappa-1}{a_l} B^l_- \delta_l^T \mathbf{b} + \boldsymbol{\nu}_-\right) \\ &= -\tfrac{b_l(\kappa-1)}{a_l}(\bar{u}_i - \bar{u}_j) + \frac{\bar{u}_i - \bar{u}_j}{\frac{a_l}{\kappa-1} + r_{ij}}(B^l_-)^T Q^{-1}\left(K^T\mathbf{b} + \boldsymbol{\nu}_- + \tfrac{b_l(\kappa-1)}{a_l} B^l_-\right) \\ &= \frac{\bar{u}_i - \bar{u}_j}{\frac{a_l}{\kappa-1} + r_{ij}}\left(-\tfrac{b_l}{a_l}(\kappa-1)\left(\tfrac{a_l}{\kappa-1} + r_{ij}\right) + \gamma^*_i - \gamma^*_j + \tfrac{b_l}{a_l}(\kappa-1) r_{ij}\right) \\ &= \frac{\bar{u}_i - \bar{u}_j}{\frac{a_l}{\kappa-1} + r_{ij}}\left(-b_l + \gamma^*_i - \gamma^*_j\right) \\ &= \frac{\bar{u}_i - \bar{u}_j}{\frac{a_l}{\kappa-1} + r_{ij}} a_l f^*_l,\end{aligned} \tag{4.16}$$

where the third and last equivalences follow from KKT conditions $Q^{-1}(K^T\mathbf{b} + \boldsymbol{\nu}_-) = \boldsymbol{\gamma}^*$ and $\gamma^*_i - \gamma^*_j = a_l f^*_l + b_l$. Proposition 4.1 relates the Lagrangian multiplier to the social cost, concluding the proof. □

In order to solve Problem 4.1 by the electrical formulation, we need to compute (4.23) for every link $l \in \mathcal{E}$ of the transportation network. The unperturbed equilibrium $\mathbf{f}^*$ is assumed to be observable and therefore given, and the potential $\mathbf{u}$ can be derived by solving a sparse linear system (**u** is harmonic with respect to $P = D^{-1}W$, as observed in Section A.2). Observe that **u** has to be computed only once. However, the computation of r_{ij} involves the solution of a linear system, and is needed for every link l, so that the solution of Problem 1 by the electrical formulation requires to solve E linear systems (see [91]). We then propose a method to *approximate* the effective resistance between a pair of neighbors that, under a suitable assumption on the sparseness of the network, does not scale with the size of the network, allowing for a more efficient solution to Problem 4.1. Before doing this, we discuss Assumption 4.1.

4.3.3 On Assumption 4.1

In the following we show that Assumption 4.1 is without loss of generality on series-parallel networks, provided that the throughput is sufficiently large.

Proposition 4.2. *Let $(\mathcal{G}, A, \mathbf{b}, \tau)$ be a routing game. If $\mathcal{G}$ is a directed series-parallel network, it exists $\overline{\tau}$ such that for every $\tau \geq \overline{\tau}$, $\mathcal{E}_+ = \emptyset$. Furthermore, if $\mathbf{b} = \mathbf{0}$, $\mathcal{E}_+ = \emptyset$ for every $\tau > 0$.*

Proof. A sufficient condition under which $\mathcal{E}_+ = \emptyset$ is that the first E components of $\mathbf{x}^* = H^{-1}\mathbf{b}$, corresponding to equilibrium link flows, are nonnegative. Indeed, since the delay functions are assumed strictly increasing, the Wardrop equilibrium is solution of strictly convex program. Then, if the flows corresponding to $\mathbf{x}^* = H^{-1}\mathbf{b}$ satisfy the constraint $\mathbf{f}^* \geq \mathbf{0}$, then $\mathbf{f}^*$ is feasible and is the unique Wardrop equilibrium, with $\boldsymbol{\lambda}^* = \mathbf{0}$ because of the complementary slackness. Hence, we look for conditions satisfying $x_l^* \geq 0$ for every $l \in \{1, \cdots, \mathrm{E}\}$. Consider an arbitrary link l with $\xi(l) = i, \theta(l) = j$. From (4.4), and from $\boldsymbol{\nu}_- = \tau\delta_o$, it follows:

$$f_l^* = -\frac{b_l}{a_l} + [KQ^{-1}K^T]_{l:}\mathbf{b} + [KQ^{-1}]_{l:}(\boldsymbol{\nu}_-).$$

As shown in the proof of Theorem 4.1, $\overline{\mathbf{u}}_- = Q^{-1}\delta_o$, where $\overline{\mathbf{u}}_-$ is the restriction of the electrical potential computed on the resistor network $\mathcal{G}_R$. Then,

$$f_l^* = -\frac{b_l}{a_l} + [KQ^{-1}K^T]_{l:}\mathbf{b} + \tau\frac{\overline{u}_i - \overline{u}_j}{a_l}.$$

If $\overline{u}_i - \overline{u}_j > 0$, then, for any $\tau \geq \overline{\tau}_l$ with

$$\overline{\tau}_l = \frac{\frac{b_l}{a_l} - [KQ^{-1}K^T]_{l:}\mathbf{b}}{\frac{\overline{u}_i - \overline{u}_j}{a_l}}$$

it holds $f_l^* \geq 0$, which in turn implies that if $\tau \geq \overline{\tau} := \max \overline{\tau}_l$, then $\mathcal{E}_+ = \emptyset$. Moreover, if the delays are linear, $\overline{u}_i - \overline{u}_j$ implies $f_l^* \geq 0$ and $\mathcal{E}_+ = \emptyset$ for any value of τ, because $\mathbf{b} = \mathbf{0}$. We have now to prove that for every link l $(\xi(l) = i, \theta(l) = j)$ of the directed transportation network, $\overline{u}_i - \overline{u}_j > 0$ in the related resistor network, which is equivalent to $u_i - u_j > 0$. Note by Ohm's law that $u_i - u_j$ is positive if and only if the current flowing from i to j is positive. Then, it suffices to show that such a current is positive. Observe that, by definition, if the transportation network is a directed series-parallel, it is a single link from o to d or it can obtained by connecting in series or in parallel two directed series-parallel networks. Thus, a series-parallel network can be reduced to a single link by recursively 1) merging two links l_1 and l_2 connected in series into a single link l_3, with $a_3 = a_1 + a_2$ (recall that the coefficients a_e correspond to resistances on the resistor network), and 2) merging two links l_1 and l_2 connected in parallel into a single link l_3, with $a_3 = a_1a_2/(a_1 + a_2)$. Moreover, observe that in both cases $\mathrm{i}_3 > 0$ if and only if $\mathrm{i}_1 > 0$ and $\mathrm{i}_2 > 0$. Indeed, in case 1) $\mathrm{i}_3 = \mathrm{i}_1 = \mathrm{i}_2$, and in case 2) $\mathrm{i}_1 = \mathrm{i}_2 a_2/a_1$ and $\mathrm{i}_3 = \mathrm{i}_1 + \mathrm{i}_2$. Obviously, when the transportation network is reduced to a single link from o to d, the flow on the unique link is positive because $\tau > 0$. Then, by applying those arguments recursively, for every link $l \in \mathcal{E}$ with $i = \xi(l), j = \theta(l)$, we get $\mathrm{i}_l > 0$, which implies by Ohm's law that $u_i - u_j > 0$. Thus, if $\tau \geq \overline{\tau}$ then $f_l^* \geq 0$ and $\mathcal{E}_+ = \emptyset$, concluding the proof. □

Remark 4.2. *Proposition 4.2 immediately implies that Assumption 4.1 is without loss of generality on directed series-parallel networks provided that* $\tau \geq \overline{\tau}$.

The next example shows that if the throughput is not sufficiently high Assumption 4.1 does not hold. However, requiring that the throughput is large does not seem a restrictive assumption for traffic applications, given the high congestion and demand of urban networks.

Example 4.1. *Consider the network in Figure 4.1, which is series-parallel. Let* $\tau = 1$, *and consider affine delay functions with* $d_1(f_1) = f_1, d_2(f_2) = f_2 + 1/2$. *The corresponding Wardrop equilibrium is*

$$f_1^* = 3/4, \quad f_2^* = 1/4, \quad \lambda_1^* = \lambda_2^* = 0.$$

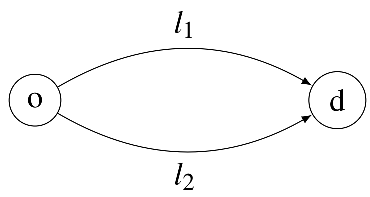


Fig. 4.1 A directed series-parallel network. If the throughput is not sufficiently large, Assumption 4.1 is not guaranteed to hold.

Turning a_1 from 1 to 1/3, we get:

$$f_1^* = 1, \quad f_2^* = 0, \quad \lambda_1^* = 0, \quad \lambda_2^* = 1/6,$$

violating Assumption 4.1. Proposition 4.2 proves that this cannot occur if τ is sufficiently large.

4.4 An approximate solution to Problem 1

As seen in the previous section, Problem 4.1 may be rephrased in terms of electrical quantities over a resistor network. However, even in this formulation the complexity of the problem scales badly because it requires to solve E linear systems whose size grows linearly with N. While **u** may be found in quasi-linear time by solving a sparse linear system (see [92]), the computational bottleneck is represented by the computation of the effective resistance between every pair of adjacent nodes of the network. To compute all the effective resistances one needs to solve E linear systems, or alternatively to invert the matrix $Q \in \mathbb{R}^{(\mathcal{N}\setminus \mathrm{d})\times(\mathcal{N}\setminus \mathrm{d})}$ or to compute the pseudoinverse of the laplacian of the resistor network (see [93] for details). For this reason, in the next subsection we propose a computationally cheaper method to approximate this quantity. The main idea of our method is that, even though the effective resistance depends on the entire network, when i and j are adjacent nodes, r_{ij} can be approximated by looking at a local portion of the network only. Such an approximation relies on cutting and shorting techniques, which have been introduce in Chapter 2. We then formulate an algorithm to approximately solve Problem 4.1 by exploiting the approximation of the effective resistance.

4.4.1 Approximating the effective resistance

Let us introduce the notion of cutting and shorting a resistor network at a distance d from a certain set of nodes. We refer to A.2 for the general notion of cutting and shorting in resistor networks.

Definition 4.2 (Cutting at distance d)**.** *Cutting a resistor network $\mathcal{G}_R$ at distance d with respect to a subset of nodes $\mathcal{S} \in \mathcal{N}$ means removing from $\mathcal{G}_R$ all the nodes at distance greater than d from $\mathcal{S}$, i.e., all nodes not belonging to $\mathcal{N}_d(\mathcal{S})$, and every link having at least one end-point in the set of the removed nodes. Let $\mathcal{G}_{\mathcal{S}}^{U_d}$ denote such a resistor network. If $\mathcal{S} = \{i, j\}$, we let for simplicity $\mathcal{G}_{ij}^{U_d}$ denote the cut network, and $r_{ij}^{U_d}$ denote the effective resistance on it.*

Definition 4.3 (Shorting at distance d)**.** *Shorting a resistor network $\mathcal{G}_R$ at distance d with respect to a subset of nodes $\mathcal{S} \in \mathcal{N}$ means shorting together all the nodes of $\mathcal{G}_R$ all the nodes at distance greater than d, i.e., all the nodes belonging to $\mathcal{N} \setminus \mathcal{N}_d(\mathcal{S})$. Let $\mathcal{G}_{\mathcal{S}}^{L_d}$ denote such a resistor network. If $\mathcal{S} = \{i, j\}$, we let for simplicity $\mathcal{G}_{ij}^{L_d}$ denote the shorted network, and $r_{ij}^{L_d}$ denote the effective resistance on it.*

We refer to Figure 4.2 for an example of these techniques applied to a regular grid. We now prove that a hierarchy between effective resistance on cut and shorted resistor network exist.

Proposition 4.3. *Let $\mathcal{G}_R$ be a resistor network and r_{ij} be the effective resistance between any two neighboring nodes i and j. Then,*

$$r_{ij}^{U_{d_1}} \geq r_{ij}^{U_{d_2}} \geq r_{ij} \geq r_{ij}^{L_{d_2}} \geq r_{ij}^{L_{d_1}}, \quad \forall\, d_2 \geq d_1 \geq 1.$$

Moreover,

$$1/D_{max} \leq r_{ij}^{L_d} \leq r_{ij}^{U_d} \leq 1/W_{ij}, \quad \forall d \geq 1, \tag{4.17}$$

where D_{max} denotes the maximal weighted degree of the resistor network, i.e., the maximal element of D.

Proof. The first set of inequality follows from Rayleigh's monotonicity laws, enunciated in Lemma A.1. The right inequality in (4.17) follows from noticing that, by Rayleigh's monotonicity laws, the effective resistance computed in the network with only nodes i and j is an upper bound for $r_{ij}^{U_1}$, and is equal to $1/W_{ij}$. The left

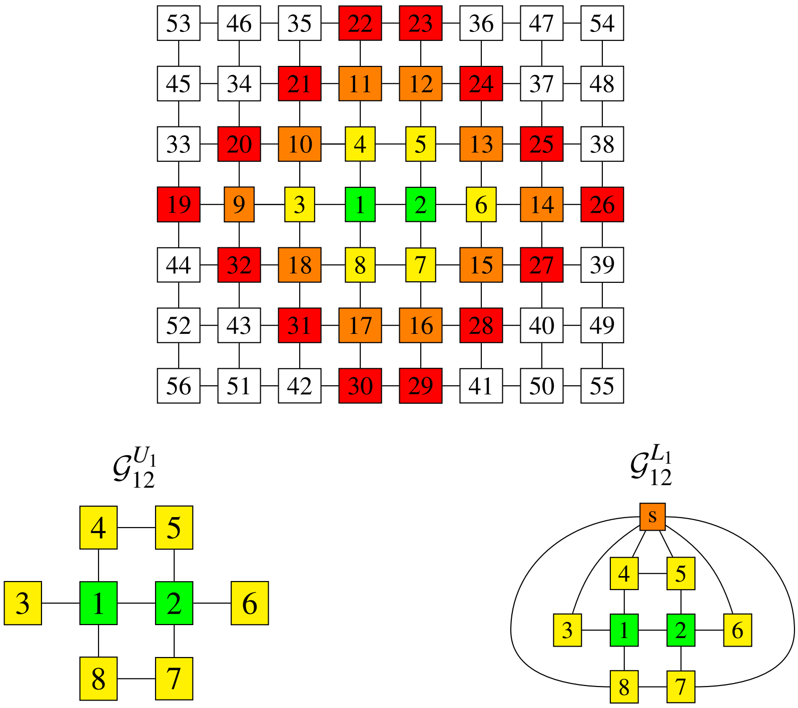


Fig. 4.2 Square grid. Above: the yellow, orange and red nodes are at distance 1, 2 and 3, respectively from the green nodes. Bottom left: cut at distance 1. Bottom right: shorted at distance 1. We stress that in the bottom right network the links connecting yellow nodes with node s do not have unitary weights.

inequality follows from noticing that the effective resistance on the network in which every node except j is shorted with i, which results in a network with only two nodes and a conductance between i and j, is no greater than D_{max} (hence, resistance no less than $1/D_{max}$) is a lower bound for $r_{ij}^{L_1}$. □

4.4.2 Our algorithm

Based on the method for approximating the effective resistance, we here propose an algorithm to approximately solve Problem 4.1. Our approach is detailed in Algorithm 1. Note that the performance of Algorithm 1 depends on the choice of the parameter d.

ALGORITHM 1:

Input: The transportation network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, the resistor network $\mathcal{G}_R = (\mathcal{N}, \mathcal{L}, W)$, the rescale parameter κ and the distance $d \geq 1$ used to approximate the effective resistance.

Output: The optimal link l^{*d} for the intervention.

Compute **u** by solving the sparse linear system

$$u_{\mathrm{o}} = 1, \quad u_{\mathrm{d}} = 0, \quad u_i = \sum_{j \in \mathcal{N}} P_{ij} u_j \quad \forall i \in \mathcal{N} \setminus \{\mathrm{o}, \mathrm{d}\};$$

for *each* $l \in \mathcal{E}$ *(let* $i = \xi(l), j = \theta(l)$*)* **do**

Construct $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$;

Compute $r_{ij}^{U_d}$ and $r_{ij}^{L_d}$ on $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$.

end

Select l^{*d} such that

$$l^{*d} \in \operatorname*{arg\,max}_{l \in \mathcal{E}} \quad \Delta C^d(l) := \frac{f_l^*(u_{i)} - u_j)}{\frac{1}{\kappa - 1} + \frac{r_{ij}^{U_d} + r_{ij}^{L_d}}{2a_l}}.$$

Specifically, the higher d is the better is the approximation of the effective resistance and the closer is the output of Algorithm 1 to achieve the minimum of Problem 1.

Theorem 4.2. *Let $\Delta C(l)$ be the social cost variation corresponding to intervention on link l with $\xi(l) = i, \theta(l) = j$ as given in Theorem 4.1, $\Delta C^d(l)$ be the social cost* *variation estimated by Algorithm 1 for a given distance $d \geq 1$, and*

$$\varepsilon_{ijd} := \frac{r_{ij}^{U_d} - r_{ij}^{L_d}}{a_l}.$$

Then,

$$\left|\frac{\Delta C(l) - \Delta C^d(l)}{\Delta C(l)}\right| \leq \frac{\varepsilon_{ijd}}{2\left(\frac{1}{\kappa-1} + \frac{r_{ij}^{U_d}+r_{ij}^{L_d}}{2a_l}\right)} \leq \frac{\varepsilon_{ijd}}{2\left(\frac{1}{\kappa-1} + \frac{1}{D_{max}\cdot a_l}\right)}$$

Furthermore,

$$\Delta C(l) \geq \iota \frac{f_l^*(u_i - u_j)}{\frac{1}{\kappa-1} + \frac{r_{ij}^{U_d}}{a_l}}. \tag{4.18}$$

Proof. First, note that

$$\begin{aligned} |\Delta C(l) - \Delta C^d(l)| &= \left| \iota \frac{f_l^*(u_i - u_j)}{\frac{1}{\kappa-1} + \frac{r_{ij}}{a_l}} - \iota \frac{f_l^*(u_i - u_j)}{\frac{1}{\kappa-1} + \frac{r_{ij}^{U_d}+r_{ij}^{L_d}}{2a_l}} \right| \\ &= \left| \iota \frac{f_l^*(u_i - u_j)}{\frac{1}{\kappa-1} + \frac{r_{ij}}{a_l}} \right| \cdot \left| \frac{\frac{r_{ij}^{U_d}+r_{ij}^{L_d}-2r_{ij}}{2a_l}}{\frac{1}{\kappa-1} + \frac{r_{ij}^{U_d}+r_{ij}^{L_d}}{2a_l}} \right|, \end{aligned}$$

Note also that

$$\frac{|r_{ij}^{U_d} + r_{ij}^{L_d} - 2r_{ij}|}{a_l} \leq \frac{|r_{ij}^{U_d} - r_{ij}| + |r_{ij} - r_{ij}^{L_d}|}{a_l} = \frac{r_{ij}^{U_d} - r_{ij} + r_{ij} - r_{ij}^{L_d}}{a_l} = \frac{r_{ij}^{U_d} - r_{ij}^{L_d}}{a_l} = \varepsilon_{ijd}.$$

Putting those two together, and using the formula for $C(l)$ obtained in Theorem 4.1, we get

$$\left|\frac{\Delta C(l) - \Delta C^d(l)}{\Delta C(l)}\right| \leq \frac{\varepsilon_{ijd}}{2\left(\frac{1}{\kappa-1} + \frac{r_{ij}^{U_d}+r_{ij}^{L_d}}{2a_l}\right)} \leq \frac{\varepsilon_{ijd}}{2\left(\frac{1}{\kappa-1} + \frac{1}{D_{max}\cdot a_l}\right)},$$

where the last inequality follows from (4.17). Eq. (4.18) follows from $r_{ij}^{U_d} \geq r_{ij}$ and from Theorem 4.1, concluding the proof. □

In the next section we provide conditions for ε_{ijd} to vanish for large distance d in the limit of infinite networks. In the rest of this section we show that the tightness of the bounds (and therefore ε_{ijd}), and their computational complexity (for a fixed d)

depend only on the local structure around the link l, and do not scale with the size of the network, under a suitable assumption.

Assumption 4.2. *Let $\mathcal{G}_R$ be a resistor network, l an arbitrary link of the network. We assume that the network is sparse in such a way that for every d the cardinality $\mathcal{N}_d(\{\xi(l), \theta(l)\})$ does not depend on* N.

Assumption 4.2 is suitable for transportation networks, because of physical constraints not allowing for the degree of the nodes to grow unlimitedly (think for instance of a square grid, where the degree of the nodes is 4 no matter what the size of the network is). Notice also that, under Assumption 4.2, N and E are proportional. Hence, from now on we refer indistinctly to N or E to denote the size of the network.

Proposition 4.4. *Let $\mathcal{G}_R = (\mathcal{N}, \mathcal{L}, W)$ be a resistor network, $\{i, j\}$ be a pair of neighboring nodes, $d \geq 1$. Then, the time complexity of the bounds and the tightness of the bounds are functions of the structure of $\mathcal{G}_R$ within distance $d+1$ from $\{i, j\}$ only. Furthermore, under Assumption 4.2 they do not depend on the size of the network.*

Proof. The cut and shorted networks are obtained by finding the neighbors within distance d and $d+1$ from $\{i, j\}$, respectively. The neighbors of a node i can be found by checking the non-zero elements of $W(i,:)$. The neighbors within distance d can be found by iterating such operation d times. Hence, the time for building the cut and the shorted network depends on the local structure, which, under Assumption 4.2, does not depend on the size of the network. Since the bounds of the effective resistance are computed on these subnetwork, their time complexity and tightness depends on local structure, which, under Assumption 4.2, is independent of the size of the network. □

Remark 4.3. *Proposition 4.4 states that under Assumption 4.2 the time complexity to approximate a single effective resistance does not scale with the size of the network for every distance d. Therefore, all the effective resistances may be approximated in linear time.* **u** *is computed via a diagonally dominant, symmetric and positive definite linear systems. The design of fast algorithms to solve this class of problem is an active field of research in the last years. To the best of our knowledge, the best algorithm has been provided in [92] and has complexity $O(M \log^k \mathrm{N} \log 1/\varepsilon)$, where ε is the tolerance error, k is a constant, and M is the number of nonzero elements* *in the matrix of the linear system. Since in our case M scales with* E*, and since* E *scales with* N *under Assumption 4.2, Algorithm 1 is quasilinear in* N*. Although our main focus is not specifically on computational complexity, but more on the methodological aspects, we remark that other approaches have a complexity at least quadratic in the number of nodes compared to the quasi-linear performance of Algorithm 1. Other approaches include:*

- *solving* E *convex programs as* (2.15)*, one for every link $l \in \mathcal{E}$ with $\tilde{a}_l$ instead of a_l;*
- *exploit the characterization* (4.5) *to compute $\gamma_0^*(l)$. This method requires the computation of the first row of $Q^{-1}(l)$ for every link $l \in \mathcal{E}$;*
- *exploit the electrical characterization and compute exactly all the r_{ij}, which requires to invert $Q \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, together with the computation of* **u** *as in Algorithm 1.*

4.5 Bound analysis

In this section we provide a characterization of the tightness of the bounds of the effective resistance between neighbors in terms of random walks over the resistor networks $\mathcal{G}_R$, $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$. We then use this characterization to provide a sufficient condition on the resistor network under which the approximation error of the bounds vanishes asymptotically as the distance d grows. To this end, we interpret the matrix P of the network as the transition matrix of the associated random walk as defined in Definition A.17, and introduce the following notation. Let $p_k(X)$, $p_k^{U_d}(X)$ and $p_k^{L_d}(X)$, denote the probability that the event X occurs, given a random walk that starts in k at the initial time, and evolves over the resistor networks $\mathcal{G}_R$, $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$, respectively. The next proposition provides a characterization for the distance between the upper and lower bound on r_{ij} in terms of probabilities of random walks over $\mathcal{G}_R$, $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$. Many of the notions needed for this proof are contained in Section A.3.

Proposition 4.5. *Let $\mathcal{G}_R = (\mathcal{N}, \mathcal{L}, W)$ be a resistor network. Based on the random walk on the resistor network, for every pair of neighboring nodes (i, j),*

$$r_{ij}^{U_d} - r_{ij}^{L_d} \leq \frac{D_{ii}}{(W_{ij})^2} \underbrace{p_i(T_{\partial_d} < T_j)}_{\text{Term 1}} \cdot \max_{g \in \partial_d} \underbrace{\left(p_g^{U_d}(T_i < T_j) - p_g^{L_d}(T_i < T_j)\right)}_{\text{Term 2}}, \tag{4.19}$$

where for simplicity of notation ∂_d denotes $\partial_d(\{i, j\})$, and $T_{\mathcal{X}}$ denotes the hitting time on a set of nodes $\mathcal{X}$, as provided in Definition A.11.

Proof. We introduce the following notation:

- The index U_d and L_d indicate that the random walk takes place over $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$, respectively. So, for instance, ${}_kG_{ij}^{U_d}$ denotes the expected number of times that the random walk on the network $\mathcal{G}_{ij}^{U_d}$, starting from i, hits j before hitting k;
- $p_i(T_j = T_{\mathcal{S}})$, with $j \in \mathcal{S}$, denotes the probability that the random walk starting from i hits the node $j \in \mathcal{S}$ before hitting any other node in $\mathcal{S}$.

By applying (A.23) to the effective resistance of a link l with $\xi(l) = i, \theta(l) = j$ in the shorted and the cut network, it follows

$$r_{ij}^{U_d} = \frac{{}_jG_{ii}^{U_d}}{D_{ii}}, \qquad r_{ij}^{L_d} = \frac{{}_jG_{ii}^{L_d}}{D_{ii}}, \tag{4.20}$$

where we recall that ${}_jG_{ii}^{U_d}$ and ${}_jG_{ii}^{L_d}$ are the expected number of visits on i, before hitting j, starting from i, of the random walk defined on $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$ respectively. The visits on i before hitting j can be divided in two disjoint sets:

- the visits before hitting j and before visiting any node in ∂_d; and
- the visits before hitting j and after visiting any node in ∂_d.

Let $G_{ii}^{<\partial_d}$ denote the expected number of visits to i, starting from i, before hitting any node in ∂_d and before hitting the absorbing node j (for simplicity of notation we omit the index j from now on). Note that $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$ differ only in the node s, which consists in the node obtained by shorting all the nodes at distance greater than d from i and j. Since s cannot be reached before hitting nodes in ∂_d before, $G_{ii}^{<\partial_d}$ is equivalent when computed on $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$. Moreover, let $G_{ii}^{U>\partial_d}$ and $G_{ii}^{L>\partial_d}$ denote the expected number of visits to i, starting from i, before hitting j but after at least one node in ∂_d has been visited in the network $\mathcal{G}_{ij}^{U_d}$ and $\mathcal{G}_{ij}^{L_d}$, respectively. Thus,

$$G_{ii}^{U_d} = G_{ii}^{<\partial_d} + G_{ii}^{U>\partial_d},$$
$$G_{ii}^{L_d} = G_{ii}^{<\partial_d} + G_{ii}^{L>\partial_d}.$$

This implies by (4.20)

$$r_{ij}^{U_d} - r_{ij}^{L_d} = \frac{G_{ii}^{U>\partial_d} - G_{ii}^{L>\partial_d}}{D_{ii}}. \tag{4.21}$$

Notice that $G_{ii}^{U>\partial_d}$ can be written as the sum over the nodes $g \in \partial_d$ of the probability, starting from i, of hitting g and going back to i without hitting j, multiplied by the expected number of visits on i starting from i, before hitting j, which is the derivative of a geometric sum [1]. Therefore,

$$\begin{aligned} G_{ii}^{U>\partial_d} &= \sum_{g\in\partial_d} \underbrace{p_i(T_g = T_{j\cup\partial_d})}_{(1)} \underbrace{p_g^{U_d}(T_i < T_j)}_{(2)} \sum_{k=1}^{\infty} k \underbrace{\left(p_i^{U_d}(T_i^+ < T_j)\right)^{k-1}}_{(3)} \underbrace{\left(1 - p_i^{U_d}(T_i^+ < T_j)\right)}_{(4)} \\ &= \frac{\sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) p_g^{U_d}(T_i < T_j)}{1 - p_i^{U_d}(T_i^+ < T_j)}, \end{aligned}$$

where:

1. probability from i of hitting g before hitting j and any other node in ∂_d;
2. probability from g of hitting i before j;
3. probability from i of hitting i $k-1$ times before hitting j;
4. probability from i of hitting j before i.

[1] Note that for $\alpha < 1$, $\sum_{k=1}^{\infty} k\alpha^{k-1}(1-\alpha) = (1-\alpha)\cdot\frac{d}{d\alpha}[\sum_{k=0}^{\infty}\alpha^k] = (1-\alpha)\frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\right) = (1-\alpha)\frac{1}{(1-\alpha)^2} = \frac{1}{1-\alpha}$.

Similarly,

$$\begin{aligned} G_{ii}^{L>\partial_d} &= \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) p_g^{L_d}(T_i < T_j) \sum_{k=1}^{\infty} k\big(p_i^{L_d}(T_i^+ < T_j)\big)^{k-1}\big(1 - p_i^{L_d}(T_i^+ < T_j)\big) \\ &= \frac{\sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) p_g^{L_d}(T_i < T_j)}{1 - p_i^{L_d}(T_i^+ < T_j)}. \end{aligned}$$

Substituting in (4.21) yields

$$r_{ij}^{U_d} - r_{ij}^{L_d} = \frac{1}{D_{ii}} \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) \left(\frac{p_g^{U_d}(T_i < T_j)}{1 - p_i^{U_d}(T_i^+ < T_j)} - \frac{p_g^{L_d}(T_i < T_j)}{1 - p_i^{L_d}(T_i^+ < T_j)} \right).$$

From (A.23), it follows

$$\begin{aligned} r_{ij}^{U_d} &= \frac{1}{D_{ii} p_i^{U_d}(T_j < T_i^+)} = \frac{1}{D_{ii}\big(1 - p_i^{U_d}(T_i^+ < T_j)\big)}, \\ r_{ij}^{L_d} &= \frac{1}{D_{ii} p_i^{L_d}(T_j < T_i^+)} = \frac{1}{D_{ii}\big(1 - p_i^{L_d}(T_i^+ < T_j)\big)}. \end{aligned}$$

Thus,

$$\begin{aligned} r_{ij}^{U_d} - r_{ij}^{L_d} &= \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) \big(p_g^{U_d}(T_i < T_j) r_{ij}^{U_d} - p_g^{L_d}(T_i < T_j) r_{ij}^{L_d}\big) \\ &= \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) \big(p_g^{U_d}(T_i < T_j) - p_g^{L_d}(T_i < T_j)\big) r_{ij}^{U_d} + \\ &+ \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) p_g^{L_d}(T_i < T_j) (r_{ij}^{U_d} - r_{ij}^{L_d}) \\ &\leq \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) \big(p_g^{U_d}(T_i < T_j) - p_g^{L_d}(T_i < T_j)\big) r_{ij}^{U_d} + \\ &+ \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) (r_{ij}^{U_d} - r_{ij}^{L_d}) \\ &= \sum_{g\in\partial_d} p_i(T_g = T_{j\cup\partial_d}) \big(p_g^{U_d}(T_i < T_j) - p_g^{L_d}(T_i < T_j)\big) r_{ij}^{U_d} + \\ &+ p_i(T_{\partial_d} < T_j)(r_{ij}^{U_d} - r_{ij}^{L_d}), \end{aligned}$$

Table 4.1 All the four cases are possible, as shown in Section 4.5.2. Term 1 $\to 0$ under the assumption that the network is recurrent, as proved in Section 4.5.1.

	Term 2 $\to 0$	Term 2 $\nrightarrow 0$
Term 1 $\to 0$	2d grid	Ring
Term 1 $\nrightarrow 0$	3d grid	Double tree

where the last inequality follows from $p_g^L(T_i < T_j) \leq 1$ and the last equality from the fact that $p_i(T_{\partial_d} < T_j) = \sum_{g \in \partial_d} p_i(T_g = T_{j \cup \partial_d})$. It follows

$$
\begin{aligned}
r_{ij}^{U_d} - r_{ij}^{L_d} &\leq \frac{\sum_{g \in \partial_d} p_i(T_g = T_{j \cup \partial_d}) \big(p_g^U(T_i < T_j) - p_g^L(T_i < T_j)\big) r_{ij}^{U_d}}{1 - p_i(T_{\partial_d} < T_j)} \\
&\leq \sum_{g \in \partial_d} p_i(T_g = T_{j \cup \partial_d}) \big(p_g^U(T_i < T_j) - p_g^L(T_i < T_j)\big) r_{ij}^{U_d} \frac{D_{ii}}{W_{ij}} \\
&\leq p_i(T_{\partial_d} < T_j) \, \cdot \, \max_{g \in \partial_d} \big(p_g^U(T_i < T_j) - p_g^L(T_i < T_j)\big) \frac{D_{ii}}{(W_{ij})^2}.
\end{aligned}
$$

where the second inequality follows from

$$1 - p_i(T_{\partial_d} < T_j) = p_i(T_j < T_{\partial_d}) \geq P_{ij} = W_{ij}/D_{ii}$$

and the last one from $r_{ij}^{U_d} \leq 1/W_{ij}$ (as shown in (4.17)) and from $p_i(T_{\partial_d} < T_j) = \sum_{g \in \partial_d} p_i(T_g = T_{j \cup \partial_d})$. □

In the next subsection we use this result to study the asymptotic behaviour of the error term $\varepsilon_{ijd} = (r_{ij}^{U_d} - r_{ij}^{L_d})/a_l$. In Section 4.5.1 we show that this error goes to zero for the class of recurrent networks (the notion of recurrent network is provided in Definition A.18). The core idea to prove this result is to show that Term 1 in (4.19) goes to zero. To generalize our analysis beyond recurrent networks, in Section 4.5.2 we study both Term 1 and 2 and provide examples showing that all combinations are possible (see Table 4.1). In particular, it is still possible that $\varepsilon_{ijd} \to 0$ for non-recurrent networks (for which Term 1 does not tend vanish) if Term 2 vanishes.

4.5.1 Recurrent networks

In this section we show that a sufficient condition under which the distance between the upper and the lower bound vanishes as the distance d goes to infinity is that the resistor network is recurrent, provided that the degree of every node is finite.

Theorem 4.3. *Let $\mathcal{G}_R$ be an infinite recurrent resistor network, and let D_{max} be finite. Then, for every link l (let $\xi(l) = i, \theta(l) = j$),*

$$\lim_{d\to+\infty}(r_{ij}^{U_d} - r_{ij}^{L_d}) = 0.$$

Proof. As stated in (A.20), a network is recurrent if and only if

$$\lim_{d+\to+\infty} p_i(T_{\partial_d} < T_j) = 0 \quad \forall i, j \in \mathcal{N}. \tag{4.22}$$

Observe that, to hit any node at distance $d+1$, the random walk starting from i has to hit at least a node at distance d. Hence, the sequence $\left\{p_i(T_{\partial_d} < T_j)\right\}_{d=1}^{+\infty}$ is non-increasing in d and the limit is well defined. Then, from (4.19), (4.22), from the fact that $0 \le p_g^{U_d}(T_i < T_j) - p_g^{L_d}(T_i < T_j) \le 1$ for every node g. From the assumptions $D_{max} < +\infty$ and $W_{ij} > 0$ (recall that i and j are adjacent nodes), it follows

$$\lim_{d\to+\infty} r_{ij}^{U_d} - r_{ij}^{L_d} \le \frac{D_{max}}{(W_{ij})^2} \lim_{d\to+\infty} p_i(T_{\partial_d} < T_j) = 0,$$

which completes the proof. □

Remark 4.4. *Theorem 4.3 implies that* $\lim_{d\to+\infty} \varepsilon_{ijd} = 0$ *on recurrent networks for every neighboring nodes i and j. Hence, by Theorem 4.2, the social cost variation corresponding to intervention on link l can be estimated with vanishing error. Observe that not only the error term ε_{ijd}, but also the relative error ε_{ijd}/r_{ij}, vanishes asymptotically, since $r_{ij} \ge 1/D_{max}$.*

Recurrence is a sufficient condition to guarantee $\lim_{d\to+\infty} \varepsilon_{ijd} = 0$, but is not necessary, as discussed in the next subsection.

4.5.2 Beyond recurrence

We here provide examples of infinite networks for all of the cases in Table 4.1. Observe that, for every link l (let $\xi(l) = i, \theta(l) = j$), the network cut at distance

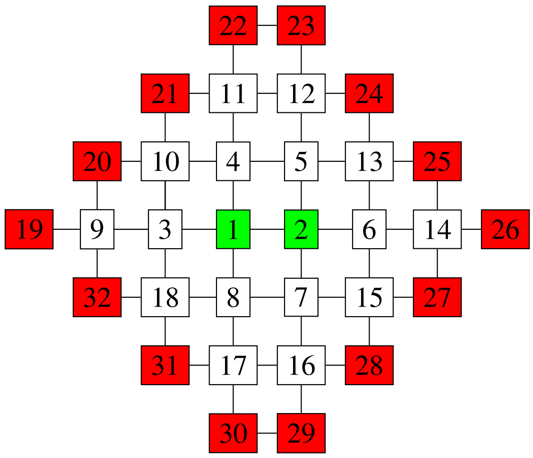


Fig. 4.3 Bidimensional square grid, cut at distance $d = 3$. The red nodes belong to ∂_d. As d grows, $p_g(T_i < T_j)$ approaches $1/2$ for each $g \in \partial_d$, because there are many short paths.

d from l and the network shorted at distance d from l differ in a node only. Let s denote such a node, which is the result of shorting all the nodes at distance greater than d from both i and j in a unique node. Intuitively speaking, our conjecture is that Term 2 in (4.19) is small when the resistor network has many short paths between I and j. Indeed, in this case, adding the node s does not largely affect probability, starting from any node in ∂_d, of hitting i before j, thus making Term 2 small. This intuition can be made more clear by the next examples.

2d grid

Consider an infinite unweighted bidimensional grid as in Figure 4.3. This network is relevant for the NDP since many transportation networks are very similar to grids. This network is recurrent, as shown in Example A.3, hence Theorem 4.3 guarantees that Term 1 and thus ε_{ijd} go to 0 for large d for every pair of adjacent nodes (i, j). Our conjecture, confirmed by numerical simulations, is that, for every node $g \in \partial_d$,

$$\lim_{d \to +\infty} p_g^{U_d}(T_i < T_j) = 1/2, \qquad \lim_{d \to +\infty} p_g^{L_d}(T_i < T_j) = 1/2.$$

Hence, this is recurrent network for which also Term 2 vanishes asymptotically.

Fig. 4.4 Left: shorted ring at distance $d = 2$. Right: cut ring at distance $d = 2$.

3d grid

Consider an infinite unweighted tridimensional grid. This network is not recurrent [94], therefore Term 1 does not go to 0 and we cannot conclude that $\varepsilon_{ijd} \to 0$ from Theorem 4.3. Nonetheless, numerical simulations show that, similarly to the bidimensional grid, for every node $g \in \partial_d$,

$$\lim_{d\to+\infty} p_g^{U_d}(T_i < T_j) = 1/2, \quad \lim_{d\to+\infty} p_g^{L_d}(T_i < T_j) = 1/2.$$

Hence, this is a non-recurrent network for which Term 2 (and therefore ε_{ijd}) vanishes as the distance grows.

Ring

Consider an infinite unweighted ring as in Figure 4.4. Consider nodes c and e as in Figure 4.4. Then,

$$p_c^{U_d}(T_i < T_j) = 1, \quad p_e^{U_d}(T_i < T_j) = 0.$$

for each d (even $d \to +\infty$), whereas,

$$p_c^{L_d}(T_i < T_j) = \frac{d}{2d+1} \xrightarrow{d\to+\infty} \frac{1}{2}, \quad p_e^{L_d}(T_i < T_j) = \frac{d+1}{2d+1} \xrightarrow{d\to+\infty} \frac{1}{2},$$

since this case is equivalent to the gambler's ruin problem [94]. Hence, Term 2 does not vanish for the ring. This is due to the fact that, on the ring, all the paths from c to j not passing in i include the node s. Still, Term 1, and therefore ε_{ijd}, vanish asymptotically by Theorem 4.3, since this network is recurrent.

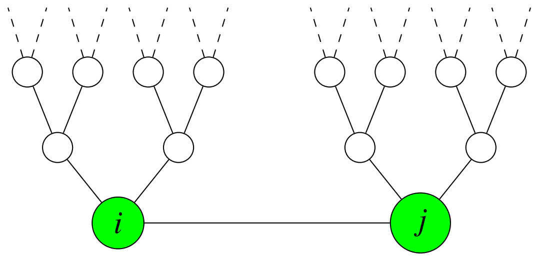


Fig. 4.5 The double tree is an infinite non-recurrent network. On this network $\lim_{d\to+\infty}\varepsilon_{ijd} = 1/3$.

Double tree network

We finally propose an infinite network for which ε_{ijd} does not converge asymptotically. This network is not relevant for traffic applications, since it admits one path only between every pair of nodes, but provides an interesting counterexample where the bounds do not converge asymptotically. The network is composed of two infinite rooted trees, starting from node i and j respectively, linked by a link l, as in Figure 4.5, and the links are assumed to have unitary conductance. It can be shown that on the double tree network the probability that the random walk, starting from i, returns on i is equal to the same quantity for a biased random walk over an infinite line (for more details we refer to the Appendix D). Since the biased random walk on a line is not recurrent (see [94])), this equivalence shows that the double tree network is non-recurrent, and Term 1 $\to 0$. Moreover, we show in Appendix D that

$$\lim_{d\to+\infty} r_{ij}^{U_d} - r_{ij}^{L_d} = \frac{1}{3},$$

thus implying that Term 2 $\nrightarrow 0$.

4.6 Simulations

In this section we illustrate numerical simulations on bidimensional grids, which provide a good proxy of transportation networks, and on a real dataset.

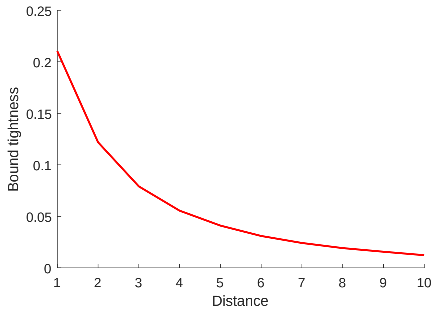


Fig. 4.6 Average relative error of the bounds on Oldenburg network as a function of distance d.

4.6.1 Infinite grids

Infinite regular grids are useful to test the performance of the bounds. Indeed, despite having an infinite number of nodes, the effective resistance between adjacent nodes can be computed exploiting the symmetric structure of the grid. We focus on the square grid, but similar arguments can be applied to any regular infinite grid. In Table 4.2 the performance of the upper and lower bounds are shown. Numerical simulations show that for every link l (let $\xi(l) = i, \theta(l) = j$),

$$\frac{r_{ij}^{U_d} - r_{ij}}{r_{ij}} = \frac{r_{ij} - r_{ij}^{L_d}}{r_{ij}} = O(1/d^2).$$

We underline that the relative errors of the bounds are symmetric only in the square grid, but they scale similarly in all the regular bidimensional grids. Observe that,

Table 4.2 Table of upper and lower bound in infinite square grid.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
$(r_{ij}^{U_d} - r_{ij})/r_{ij}$	$1/5$	0.0804	0.0426	0.0262	0.0178
$(r_{ij} - r_{ij}^{L_d})/r_{ij}$	$1/5$	0.0804	0.0426	0.0262	0.0178

despite the network being infinite, even at $d = 5$, the upper and the lower bounds give a good estimation of the true effective resistance.

4.6.2 Simulations on a real transportation network

In this section we present the performance of the cutting and shorting techniques on the traffic network of the city of Oldenburg ([95]). The network is composed of 6105 nodes and 7035 links, and its diameter is 104. The network is assumed to be unweighted, with $a_l = 1$ for every link l (let as usual $\xi(l) = i, \theta(l) = j$). The average relative error of the bounds, i.e.,

$$AT_d := \frac{1}{\mathrm{E}} \sum_{l \in \mathcal{E}} \frac{r_{ij}^{U_d} - r_{ij}^{L_d}}{r_{ij}}$$

is shown in Table 4.3 and Figure 4.6. Even for this network, the error of the bounds

Table 4.3 Table of the average relative error of the bounds at distance d.

	d=1	d=2	d=3	d=4	d=5	d=6	d=7	d=8	d=9	d=10
AT_d	0.21	0.12	0.079	0.056	0.041	0.031	0.024	0.019	0.016	0.012

decreases quickly, compared to the diameter of the network.

4.7 Additional considerations

4.7.1 Adding a link

In the last section of this chapter we present an analogous problem for the case of adding a link instead of improving an existing one. A similar interpretation in terms of resistor networks can be obtained. However, since the two terminal nodes of the new link can be in principle be distant in the original network, the approximation methods of the effective resistance do not apply. As before, our characterization works under the assumption that the set of the used links does not change due to the intervention, except for the new link. Let l be the candidate link to constructed, let $\mathcal{G}(l)$ be the network that is obtained by adding the link l to the network $\mathcal{G}$, and let $A(l), \mathbf{b}(l), \mathbf{x}(l)$, the corresponding quantities. Our characterization works under the following assumption.

Assumption 4.3. *Let $\mathcal{E}_+(l)$ be the set of links e for which $\lambda_e^*(l) > 0$ in the Wardrop equilibrium of $(\mathcal{G}, A(l), \mathbf{b}(l), \tau)$. We assume that $\mathcal{E}_+(l) = \mathcal{E}_+$ for all $l \in \mathcal{E}$.*

Theorem 4.4. *Let $(\mathcal{G}, A, \mathbf{b}, \tau)$ be a routing game. Consider a candidate new link l, and the game $(\mathcal{G}(l), A(l), \mathbf{b}(l), \tau)$ corresponding to addition of link l. Assume that Assumption 4.1 for the link l holds. By letting $\xi(l) = i, \theta(l) = j$, the social cost variation corresponding to the addition of link l is*

$$\Delta C(l) := C(\mathbf{f}^*) - C(\mathbf{f}^*(l)) = \iota \frac{(u_i - u_j)(\gamma_i^* - \gamma_j^* - b_l)}{a_l + r_{ij}}, \tag{4.23}$$

where ι is a constant independent of l, r_{ij} is the effective resistance between nodes i and j on the related resistor network $\mathcal{G}_R$, and $\mathbf{u} \in \mathbb{R}^{\mathcal{N}}$ denotes the potential over the nodes of $\mathcal{G}_R$ when the boundary conditions $u_{\mathrm{o}} = 1$ and $u_{\mathrm{d}} = 0$ are imposed.

Proof. The proof follows the steps of the proof of Theorem 4.1. Note that adding link l corresponds to a rank-1 perturbation of Q. In particular,

$$Q(l) = Q + \frac{1}{a_l} B_-^l (B_-^l)^T, \tag{4.24}$$

where B_-^l denotes the $l-$th column of B_-. Thus, by Sherman-Morrison formula,

$$Q(l)^{-1} = Q^{-1} - \frac{Q^{-1} B_-^l (B_-^l)^T Q^{-1}}{a_l + (B_-^l)^T Q^{-1} B_-^l}. \tag{4.25}$$

Also, for a link l with $\xi(l) = i, \theta(l) = j$, it holds:

$$K(l) = \begin{pmatrix} K \\ (B_-^l)^T / a_l \end{pmatrix}. \tag{4.26}$$

From $\mathbf{x}^* = H^{-1}\mathbf{y}$, it follows

$$\begin{aligned} \boldsymbol{\gamma}^* - \boldsymbol{\gamma}^*(l) &= Q^{-1}(K^T \mathbf{b} + \boldsymbol{\nu}_-) - \left(Q^{-1} - \frac{Q^{-1} B_-^l (B_-^l)^T Q^{-1}}{a_l + (B_-^l)^T Q^{-1} B_-^l} \right) \left(K^T \mathbf{b} + \frac{b_l}{a_l} B_-^l + \boldsymbol{\nu}_- \right) \\ &= -\frac{b_l}{a_l} Q^{-1} B_-^l + \frac{Q^{-1} B_-^l (B_-^l)^T Q^{-1}}{a_l + (B_-^l)^T Q^{-1} B_-^l} \left(K^T \mathbf{b} + \frac{b_l}{a_l} B_-^l + \boldsymbol{\nu}_- \right) \end{aligned} \tag{4.27}$$

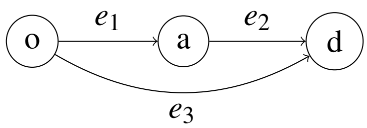


Fig. 4.7 An example to show that the social cost is not submodular.

We now follow same steps as in the proof of Theorem 4.1, i.e., we substitute $r_{ij} = (B^l_-)^T Q^{-1} B^l_-$ and $\overline{\mathbf{u}}_- = Q^{-1}\delta_o$. Thus,

$$\begin{aligned}\gamma^*_o - \gamma^*_o(l) &= -\frac{b_l}{a_l}(\overline{u}_i - \overline{u}_j) + \frac{\overline{u}_i - \overline{u}_j}{a_l + r_{ij}} (B^l_-)^T Q^{-1}\left(K^T \mathbf{b} + \frac{b_l}{a_l} B^l_- + \boldsymbol{\nu}_-\right) \\ &= \frac{\overline{u}_i - \overline{u}_j}{a_l + r_{ij}} \left(-\frac{b_l}{a_l}\left(a_l + r_{ij}\right) + \gamma^*_i - \gamma^*_j + \frac{b_l}{a_l} r_{ij}\right) \\ &= \frac{\overline{u}_i - \overline{u}_j}{a_l + r_{ij}} \left(-b_l + \gamma^*_i - \gamma^*_j\right)\end{aligned} \tag{4.28}$$

where the third equivalence follows from KKT conditions $Q^{-1}(K^T\mathbf{b} + \boldsymbol{\nu}_-) = \boldsymbol{\gamma}^*$. Proposition 4.1 relates the Lagrangian multiplier to the social cost, concluding the proof.

□

Remark 4.5. *A corollary of Theorem 4.1 is that Braess' paradox cannot occur if the network is series-parallel, i.e., improving a link cannot have a negative impact on the social cost. To prove this, we refer to the proof of Proposition 4.2, where it is proved that if the original directed network is series-parallel and $\xi(l)$ and $\theta(l)$ are the end-nodes of a link $l \in \mathcal{E}$, then the electrical potential on the related resistor network is decreasing along the link, i.e., it satisfies $u_{\xi(l)} - u_{\theta(l)} > 0$. Note that this remark on Braess' paradox holds under the assumption that delay functions are affine and the support of Wardrop equilibrium is not modified by the intervention. A more general result on non-emergence of Braess' paradox in series-parallel networks is already established in a more general setting in [96, Theorem 1].*

4.7.2 On submodularity of the objective function

Our methods work for a single intervention, since a single intervention may be seen as a rank-1 perturbation of the KKT conditions. It is natural then to ask whether our results can be generalized. A typical way to proceed is to show that the objective function (to be maximized) is submodular, and in such a case one can apply the greedy algorithm with performance guarantees [97, 98]. Such an approach is used for instance in [99] in an optimal target problem, with the goal of maximizing the spread of influence through social networks. Unfortunately, we here show by an example that our objective function is not in general submodular.

Definition 4.4. *Let $\mathcal{X}$ be a set of elements, and consider a function $g : 2^{\mathcal{X}} \to \mathbb{R}$. g is submodular if for every pair of sets $\mathcal{S}, \mathcal{T}$ such that $\mathcal{S} \subset \mathcal{T} \subset \mathcal{X}$ and additional element $x \in \mathcal{X}$,*

$$g(\mathcal{S} \cup \{x\}) - g(\mathcal{S}) \geq g(\mathcal{T} \cup \{x\}) - g(\mathcal{T}).$$

Example 4.2. *Consider the network in Figure 4.7, let $\tau = 3$, and consider delay functions*

$$d_1(f_1) = f_1 + 1, \quad d_2(f_2) = 2f_2 + 2, \quad d_3(f_3) = f_3 + 4. \tag{4.29}$$

The Wardrop equilibrium and the optimal lagrangian multiplier (by setting $\gamma_{\mathrm{d}}^* = 0$) are respectively

$$\mathbf{f}^* = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{\gamma}^* = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}. \tag{4.30}$$

We now consider an intervention on link e_1, with rescaling factor $\kappa = 2$, which leads to $d_1(f_1) = f_1/2 + 1$. The corresponding solution is

$$\mathbf{f}(\{e_1\})^* = \begin{pmatrix} 8/7 \\ 8/7 \\ 13/7 \end{pmatrix}, \quad \boldsymbol{\gamma}(\{e_1\})^* = \begin{pmatrix} 41/7 \\ 30/7 \\ 0 \end{pmatrix}. \tag{4.31}$$

Intervening on link e_2 leads to

$$\mathbf{f}(\{e_2\})^* = \begin{pmatrix} 4/3 \\ 4/3 \\ 5/3 \end{pmatrix}, \quad \boldsymbol{\gamma}(\{e_2\})^* = \begin{pmatrix} 17/3 \\ 10/3 \\ 0 \end{pmatrix}. \tag{4.32}$$

Finally, intervention on both e_1 and e_2 leads to

$$\mathbf{f}(\{e_1, e_2\})^* = \begin{pmatrix} 8/5 \\ 8/5 \\ 7/5 \end{pmatrix}, \quad \boldsymbol{\gamma}(\{e_1, e_2\})^* = \begin{pmatrix} 27/5 \\ 18/5 \\ 0 \end{pmatrix}. \tag{4.33}$$

Let $\mathcal{T} = \{e_1\}$, $\mathcal{S} = \emptyset$, which satisfies $\mathcal{S} \subset \mathcal{T}$. Our objective function to be maximized is $-\gamma_o^*$, since γ_o^* is equivalent to the social cost unless for a negative proportionality factor. Note that

$$\gamma_o^*(\mathcal{T}) - \gamma_o^*(\mathcal{T} \cup \{e_2\}) = 41/7 - 27/5 = 16/35, \tag{4.34}$$

$$\gamma_o^*(\mathcal{S}) - \gamma_o^*(\mathcal{S} \cup \{e_2\}) = 6 - 17/3 = 1/3. \tag{4.35}$$

The last two equations together show that an intervention on link e_2 after intervention on e_1 leads to a greater social cost gain than an intervention on e_2 only, and showing thus that our objective function is not submodular. Such a result is not surprising: indeed, after intervention on e_1 more users will use route (e_1, e_2), thus making more users positively affected by a second intervention on link e_2.

4.8 Conclusions

In this work we study a discrete network design problem, where a single link can be improved. We reformulate the problem in terms of electrical quantities, in particular in terms of the effective resistance of the link, and provide a method to approximate the effective resistance of a link by performing only local computations. Both the tightness and the computational complexity of our bounds do not depend on the size of the network, but on the local structure of the network only. Based on the electrical formulation and our approximation method for the effective resistance we propose an efficient algorithm to solve the network design problem in approximation. We then study the optimality of our algorithm in the limit of infinite resistor networks, and prove analytically that if the network is recurrent the approximation error vanishes asymptotically. Finally, we show by simulations that the approximation of the effective resistance achieves good performance even for small distances, both on infinite square grids and on a real dataset.

While our theoretical results characterize only the asymptotic performance of our algorithm, an interesting direction for the future is a deeper analysis on the performance for finite distances d. Finding an efficient algorithm to select the best new link to add in a transportation network exploiting the electrical characterization of the social cost variation is another interesting open issue. Future research lines also include extending the analysis to the case of multiple interventions, and the relaxation of some assumptions, e.g., the single origin and destination, and the assumption that the set of used links is not affected by the intervention.

Chapter 5

Conclusion

5.1 Summary and contribution

In this dissertation we study two different problems related to routing games on transportation networks. We first study in Chapter 3 the stability of Wardrop equilibria in heterogeneous games under evolutionary dynamics, focusing in particular on the logit dynamics. Specifically, we characterize the behaviour of the logit dynamics in the large noise and small noise regimes, and provide sufficient conditions on the network topology and on the features of the Wardrop equilibria under which the Wardrop equilibria of the game are asymptotically stable under the logit dynamics. We then propose in Chapter 4 a network design problem where the planner can improve one link of the network, with the goal of minimizing the total travel time experienced on the network at equilibrium. We show that the problem can be rephrased via an electrical formulation, and exploit such a characterization to propose an efficient algorithm to select the optimal link to improve.

In the first part of the dissertation we provide the model's description. We define multigraphs to model transportation networks, and formalize routing games to model user's choices on transportation networks in a game-theoretic framework. We trace a fundamental distinction between homogeneous routing games, where users takes decisions based on identical user cost functions, and heterogeneous routing games, that take into account the heterogeneity of users in the user cost functions, e.g., due to different routing apps or different trade-offs between time and money.

In the second part of the dissertation, we propose and analyse two problems and establish novel theoretical results. In particular, in Chapter 3 we investigate the stability of Wardrop equilibria under evolutionary dynamics in heterogeneous routing games. Specifically, we prove the following results:

- for every heterogeneous routing game, the set of the fixed points of the corresponding logit dynamics approaches a subset (called *limit equilibria*) of the Wardrop equilibria of the game in the limit of vanishing noise, and show that strict equilibria are always limit equilibria of the game;
- for every heterogeneous routing game on series of parallel multigraphs, every exact target monotone dynamics (which include the logit dynamics) admits a globally asymptotically stable fixed point;
- for every heterogeneous routing game, in the large noise regime the logit dynamics admits a globally asymptotically stable fixed point on every multigraph;
- every strict equilibrium of heterogeneous routing games is locally asymptotically stable under the logit dynamics in the limit of vanishing noise.

In Chapter 4 we instead study network design problems. Our contribution is the following:

- we show that, under a suitable regularity assumption on the Wardrop equilibrium of the game, which states that the support of the Wardrop equilibrium does not vary with an intervention, the social cost variation corresponding to intervention on a certain link may be rephrased in terms of electrical quantities computed on a related resistor network, in particular in terms of the effective resistance between the endpoints of the considered link;
- we show that our regularity assumption is without loss of generality if the network is series-parallel and the inflow to the network is sufficiently large;
- we propose a method to approximate the effective resistance between two neighbouring nodes of a resistor network, based on the local structure of the network around the pair of nodes. We then provide theoretical guarantees on the asymptotic tightness of our bounds on recurrent networks with an infinite countable set of nodes;

- we exploit the electrical characterization of the network design problem, and our approximation method for the effective resistance to propose an algorithm that selects the optimal link for the intervention performing only local computations, with a time-complexity that scales quasi-linearly with the size of the network;
- we conduct numerical simulations on synthetic and real transportation networks to validate our methods.

5.2 Future research

Future research lines mainly rely on the generalization of the obtained the results to more general settings and the relaxation of restrictive assumptions. On network design problems, future research lines include:

- considering multiple interventions, and extend the analysis to the case of multiple origin-destination pairs, and to heterogeneous routing games;
- the relaxation of the regularity assumption on the support of the Wardrop equilibrium not varying after the intervention;
- the refinement of obtained results, e.g., integrating the asymptotic results on the tightness of effective resistance approximation with theoretical results that characterize the approximation error of the effective resistance for small distances.

About the stability of evolutionary dynamics in routing games, we identify the following main directions for the future:

- concerning the logit dynamics, the characterization of the asymptotic behaviour of the dynamics on arbitrary multigraphs, e.g., to understand whether the logit dynamics always converges to fixed points on arbitrary multigraphs;
- a complete characterization of the set of the limit equilibria, to understand in case of multipla Wardrop equilibria which equilibria are selected by the dynamics;

- investigating the behaviour of other evolutionary dynamics;
- a generalization of the results to the case of multiple-origin destination pairs, and integrating in the model the physical dynamics of mass, as done in [68, 69] for homogeneous routing games.

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Appendix A

Additional notions on networks

We start by introducing undirected unweighted graphs in Section A.1. Then, we define resistor networks in Section A.2. In Section A.3, we introduce and analyse random walks on networks, and investigate relations between resistor networks and random walks on networks.

A.1 Graphs

We start defining undirected weighted graphs.

Definition A.1. *An undirected weighted graph is a triple* $(\mathcal{N}, \mathcal{L}, W)$*, where*

- $\mathcal{N}$ *is the set of nodes, whose cardinality is* $\mathrm{N} := |\mathcal{N}|$*;*
- $\mathcal{L}$ *is the set of links, whose cardinality is* $\mathrm{L} := |\mathcal{L}|$*. Links are non-ordered pairs of nodes* $\{i, j\}$*,* $i \neq j$*.*
- $W \in \mathbb{R}_+^{\mathcal{N} \times \mathcal{N}}$ *is the symmetric adjacency matrix, i.e.,* $W = W^T$*.*

A link $\{i, j\}$ has to be meant as a symmetric connection between nodes i and j, whose strength is measured by W_{ij}, with $W_{ij} = 0$ if $\{i, j\} \notin \mathcal{L}$, and $W_{ij} \neq 0$ otherwise. The notion of path, two-terminal graph and network flow optimization problems on graphs may be generalized from Chapter 2 for undirected weighted graphs. Starting from W one can define the *degree distribution* $\mathbf{w} \in \mathbb{R}_+^{\mathcal{N}}$, the *degree* *matrix* $D \in \mathbb{R}_+^{\mathcal{N}\times\mathcal{N}}$, the *Laplacian* $\Delta \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, the *normalized adjacency matrix* $P \in \mathbb{R}_+^{\mathcal{N}\times\mathcal{N}}$, and the *total degree* $w \in \mathbb{R}_+$ of the network, which are respectively

$$\mathbf{w} := W\mathbf{1}, \quad D := \operatorname{diag}(\mathbf{w}), \quad \Delta := D - W, \quad P := D^{-1}W, \quad w := \mathbf{w}^T\mathbf{1}. \tag{A.1}$$

Note that the normalized adjacency matrix P is entry-wise positive and row-stochastic, i.e., $P\mathbf{1} = \mathbf{1}$.

Definition A.2 (Distance)**.** *The distance between two nodes i and j, denoted by* $\operatorname{dist}(i,j)$*, is*

$$\operatorname{dist}(i,j) = \min\{d \geq 0 : (W^d)_{ij} > 0\}.$$

A graph is connected if for any $i, j \in \mathcal{N}$, there exists $d > 0$ such that $(W^d)_{ij} > 0$.

Definition A.3 (Neighborhood)**.** *Given a network and a subset of nodes $\mathcal{K} \subset \mathcal{N}$, The neighborhood of $\mathcal{K}$ at distance d, denoted by $\mathcal{N}_d(\mathcal{K})$, is*

$$\mathcal{N}_d(\mathcal{K}) := \{j \in \mathcal{N} : \exists\, k \in \mathcal{K} \text{ s.t. } \operatorname{dist}(j,k) \leq d\}. \tag{A.2}$$

We also let $\partial_d(\mathcal{K})$ (with $d > 0$) denote the set of nodes belonging to $\mathcal{N}_d(\mathcal{K}) \setminus \mathcal{N}_{d-1}(\mathcal{K})$. Intuitively, $\partial_d(\mathcal{K})$ contains all the nodes that are at distance d from at least a node $k \in \mathcal{K}$, and at distance no less than d from every other node in $\mathcal{K}$. We now introduce as examples the 2d square grids.

Example A.1 (Undirected grids)**.** *The 2d square grid is an undirected network in which every node n is characterized by a position (x_n, y_n), and $l = \{a,b\} \in \mathcal{L}$ if and only if either i) $x_a = x_b \pm 1$, and $y_a = y_b$ or ii) $x_a = x_b$ and $y_a = y_b \pm 1$. In Figure A.1 it is highlighted the neighborhood at distance d of an arbitrary node o of the grid, and the boundary $\partial_d(o)$ of such a set. Such a construction may be generalized to define k-dimensional grids.*

A.2 Resistor networks

We start by defining harmonic functions, which constitute a key tool to study resistor networks. Then, we introduce resistor networks and study electrical current in resistor networks by means of a network flow optimization problem. For a complete reference on these topics we refer to [94, 91, 100].

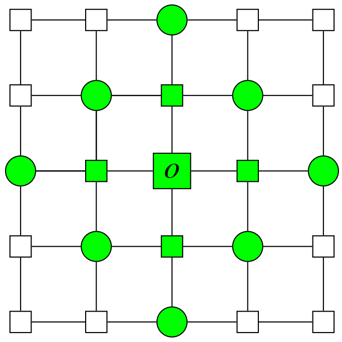

Fig. A.1 The bidimensional square grid. The green nodes belong to $\mathcal{N}_2(o)$, the circular nodes belong to $\partial_2(o)$

.

Definition A.4. *Let P be the normalized adjacency matrix of a graph $(\mathcal{N},\mathcal{L},W)$ and $\mathcal{B} \subseteq \mathcal{N}$ a subset of nodes called* boundary. *A* harmonic function *with respect to P is a function $h : \mathcal{N} \to \mathbb{R}$ with boundary conditions $h_{\mathcal{B}}$, and*

$$h_i = \sum_{j \in \mathcal{N}} P_{ij} h_j \quad \forall i \in \mathcal{N} \setminus \mathcal{B}. \tag{A.3}$$

In other words, a harmonic function satisfies the property that for any node $i \in \mathcal{N} \setminus \mathcal{B}$, the value h_i is the weighted average of the value of the function in the neighbooring nodes of i, whereas the value on the boundary $\mathcal{B}$ is given a priori. The next proposition states that given P and the boundary conditions, harmonic functions are unique.

Proposition A.1 ([94])**.** *Let $h : \mathcal{N} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$ two harmonic functions with respect to P, with same boundary $\mathcal{B} \subseteq \mathcal{N}$ and equivalent boundary conditions $h_{\mathcal{B}} = g_{\mathcal{B}}$. Then, $h_k = g_k$ for every $k \in \mathcal{N}$.*

With this tool in mind, we can now study resistor networks.

Definition A.5 (Resistor network)**.** *A* resistor network *is a undirected weighted graph $\mathcal{G}_R = (\mathcal{N},\mathcal{L},W)$ with weights W_{ij} corresponding to conductance between i and j.*

We usually denote resistor networks by $\mathcal{G}_R$. Every undirected weighted graph may be interpreted as a resistor network. Without loss of generality, throughout this dissertation we shall consider connected resistor networks. Indeed, if a resistor network is composed of two or more connected components, one can without loss of generality consider every connected component as an independent network.

Given a resistor network and an exogenous flow $\boldsymbol{\nu}$, the electrical current $\mathbf{i}$ flowing along the links of the network is the result of the following network flow optimization:

$$\mathbf{i}^* = \underset{\substack{\mathbf{i}\in\mathbb{R}^{\mathcal{L}} \\ B\mathbf{i}=\boldsymbol{\nu}}}{\arg\min} \sum_{e\in\mathcal{L}} \frac{r_e \mathrm{i}_e^2}{2}, \tag{A.4}$$

where $r_e = 1/\omega_e$ is the resistance of the link. The term $r_e\mathrm{i}_e^2$ represents the power dissipated on link e, so that (A.4) is equivalent to requiring that $\mathbf{i}^*$ is the feasible flow that minimizes the total power dissipation. Note that the solution $\mathbf{i}^*$ is unique, since the objective function is strictly convex. We restrict our analysis to exogenous flows in the form $\boldsymbol{\nu} = I(\delta_a - \delta_b)$, which means that the current is injected in node a and exits the system via node b. Using the techniques presented in Section 2.3 for network flow optimization problems, one can prove that the optimal lagrangian multiplier $\boldsymbol{\gamma}^*$ is harmonic with respect to P, where P is the normalized adjacency matrix of the resistor network. Morevoer, by using Ohm's laws, one can additionally shows that $\boldsymbol{\gamma}^*$ corresponds to the electrical potential measured on the network under exogenous flow $\boldsymbol{\nu}$. We shall denote such a potential by $\mathbf{u} \in \mathbb{R}^{\mathcal{N}}$, and refer to [94] for more details. We now introduce the notion of effective resistance, which proves very important for traffic applications, as shown in Chapter 4.

Definition A.6 (Effective resistance)**.** *Given a resistor network and an exogenous flow $\boldsymbol{\nu} = I(\delta_a - \delta_b)$, the effective resistance is the ratio (independent of I)*

$$r_{ab} := \frac{u_a - u_b}{I}.$$

Remark A.1. *For coherence with the network flow optimization notation, we presented $\mathbf{i}^*$ as the flow resulting under an exogenous flow $I(\delta_a - \delta_b)$. However, in circuit theory it is more common to fix the potential on a certain set of nodes instead of the exogoneous flow. In particular, an equivalent flow $\mathbf{i}^*$ is obtained by fixing the potential on the nodes a and b, with $u_a = I \cdot r_{ab}, u_b = 0$, instead of imposing the exogenous flow.*

We now introduce the notion of shorting and cutting resistor networks.

Definition A.7 (Cutting and shorting)**.** *Consider a resistor network and a link $e = \{i, j\} \in \mathcal{L}$.* Cutting *the link e means removing the link e, which yields a transformation from*

$$\mathcal{G}_{R_1} = (\mathcal{N}, \mathcal{L}, W_1) \to \mathcal{G}_{R_2} = (\mathcal{N}, \mathcal{L} \setminus e, W_2), \tag{A.5}$$

with $W_2 = W_1 - (W_1)_{ij}\delta_i\delta_j^T - (W_1)_{ij}\delta_j\delta_i^T$. Shorting *two nodes i, j of a network has two equivalent representations:*

- *merging the two nodes i, j in a unique node k, with $W_{kx} = W_{xk} = W_{ix} + W_{jx}$ for every node x, or equivalently*
- *mantaining the same set of nodes, while adding an infinite conductance between the pair of shorted nodes.*

Note that if $\mathcal{G}_{R_2}$ is obtained by cutting link e in $\mathcal{G}_{R_1}$, then $W_2 \preccurlyeq W_1$. Instead, if $\mathcal{G}_{R_2}$ is obtained by shorting two nodes i and j of $\mathcal{G}_{R_1}$, using the second representation the resulting network satisfies $W_1 \preccurlyeq W_2$.

Remark A.2. *The notion of effective resistance may be generalized for set of nodes by using the notion of shorting. Let $\mathcal{A}, \mathcal{B} \notin \mathcal{N}$ denote two disjoints subsets of nodes, and consider the network obtained by shorting all the nodes in $\mathcal{A}$ in a unique node a, and all the nodes in $\mathcal{B}$ in a unique node b. Then, the effective resistance between $\mathcal{A}$ and $\mathcal{B}$ is the ratio*

$$r_{ab} := \frac{u_a - u_b}{I},$$

where $\mathbf{u}$ is the potential distribution obtained on the shorted network under the exogenous flow $\boldsymbol{\nu} = I(\delta_a - \delta_b)$.

The next lemma establish a monotonicity property between effective resistance and conductance matrices.

Lemma A.1 (Rayleigh's monotonicity laws [94])**.** *Let $\mathcal{G}_{R_1} = (\mathcal{N}, \mathcal{L}_1, W_1)$ and $\mathcal{G}_{R_2} = (\mathcal{N}, \mathcal{L}_2, W_2)$ be two resistor networks with the same set of nodes and weight matrices satisfying $W_1 \preccurlyeq W_2$. Let $r_{1_{ij}}$ and $r_{2_{ij}}$ denote the effective resistance between an arbitrary pair of nodes $\{i, j\}$ in $\mathcal{G}_{R_1}$ and $\mathcal{G}_{R_2}$ respectively. Then, $r_{1_{ij}} \geq r_{2_{ij}}$.*

Remark A.3. *An immediate consequence of Rayleigh's monotonicity laws is that shorting and cutting respectively decreases and increases the effective resistance between an arbitrary pair of nodes, as the next example confirms.*

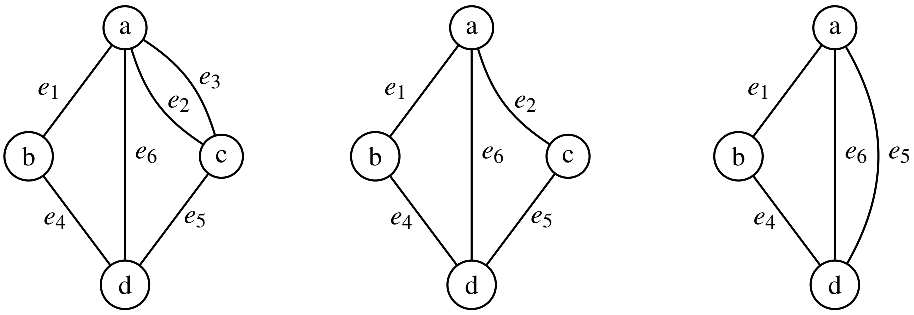

Fig. A.2 From left to right: the original network, the network cut in e_3, and the network with a and c shorted.

Example A.2. *We consider the left network in Fig. A.2, a cut version of the network and a shorted version of the network, with all the links carrying a unitary conductance. Let r_{bd}, r^u_{bd} and r^l_{bd} denote the effective resistance between nodes b and d in the three networks, respectively. By simple computations, one can show that*

$$r_{bd} = 8/13, \quad r^u_{bd} = 5/8, \quad r^l_{bd} = 3/5,$$

confirming that $r^l_{bd} \leq r_{bd} \leq r^u_{bd}$, in accordance with Rayleigh's monotonicity laws.

A.3 Random walks on networks

We start by defining the notion of discrete-time homogeneous Markov chains (MC), and then introduce random walks on networks. Then, we provide useful notions on random walks, and discuss how these notions may be extended to networks with an infinite countable node set. Afterwards, we use results from harmonic functions theory to establish connections between resistor networks and random walks over the associated network. Let $t \in \mathbb{N}$ denote time. A Markov chain $x(t) \in \mathcal{X}$ is a discrete-time process with no memory, i.e.,

$$\mathbf{P}\{x(t) = i_t | x(t-1) = i_{t-1}, \cdots, x(0) = i_0\} = \mathbf{P}\{x(t) = i_t | x(t-1) = i_{t-1}\},$$

where $\mathbf{P}\{\cdot\}$ is the probability that the event $\cdot$ occurs, $\mathcal{X}$ denotes the state space of the Markov chain and $i_0, ..., i_t \in \mathcal{X}$. We let N denote the cardinality of $\mathcal{X}$. We restrict our analysis to homogeneous-time Markov chains, whose transition probabilities do not depend on time. Let $P \in \mathbb{R}_+^{\mathcal{N} \times \mathcal{N}}$ denote the probability transition matrix of the

MC, which is required to be row-stochastic. The element P_{ij} denotes the probability of transition from i to j, i.e.,

$$P_{ij} := \mathbf{P}\{x(t+1) = j | x(t) = i\}. \tag{A.6}$$

Let $\boldsymbol{\pi}(t) \in \mathbb{R}_+^{\mathcal{N}}$ denote the probability distribution of the Markov chain at time t, whose element $\pi_i(t) := \mathbf{P}\{x(t) = i\}$ is the probability that the Markov chain is in state i at time t. The evolution of $\boldsymbol{\pi}$ in compact form reads

$$\boldsymbol{\pi}(t+1) = P^T \boldsymbol{\pi}(t). \tag{A.7}$$

A MC is fully characterized by its state space $\mathcal{X}$, a transition probability matrix P, and a non-negative initial distribution $\boldsymbol{\pi}(0)$ such that $\boldsymbol{\pi}(0)^T\mathbf{1} = 1$. Note that the row-stochasticity of P is required to preserve the normalization of $\boldsymbol{\pi}(t)$, since for every row-stochastic P

$$\mathbf{1}^T \boldsymbol{\pi}(t) = \mathbf{1}^T (P^T)^t \boldsymbol{\pi}(0) = \mathbf{1}^T \boldsymbol{\pi}(0) = 1. \tag{A.8}$$

We now introduce some important notions on Markov chains.

Definition A.8 (Invariant distribution). *The distribution $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi} = P^T \boldsymbol{\pi}$ and $\boldsymbol{\pi}^T\mathbf{1} = 1$ is called* invariant distribution.

Note that the invariant distribution of a MC is unique because of Perron-Frobenius theory on entry-wise non-negative matrices [101–103].

Definition A.9 (Reversible MC). *A MC is said to be* reversible *with respect to a distribution $\boldsymbol{\pi}$ if for every $i, j \in \mathcal{X}$,*

$$\pi_i P_{ij} = \pi_j P_{ji}. \tag{A.9}$$

Note that the unique normalized distribution satisfying (A.9) is the invariant distribution.

Definition A.10 (Irreducible MC). *A Markov chain is* irreducible *if for any i, j there exists $d \geq 0$ such that $(P^d)_{ij} \neq 0$.*

Note that the notion of irreducibility in Markov chain theory resembles the notion of strongly connectedness in graph theory. This relation shall be made clearer when studying random walks over networks. We now introduce some random variables that are of interest within this dissertation.

Definition A.11 (Hitting time). *The hitting time $T_{\mathcal{I}}$ is a random variable that indicates the first time $t \geq 0$ in which the MC hits $\mathcal{I} \subseteq \mathcal{X}$, i.e.,*

$$T_{\mathcal{I}} := \min\{t \geq 0 : x(t) \in \mathcal{I}\} \tag{A.10}$$

Definition A.12 (Return time). *The return time $T_{\mathcal{I}}^{+}$ is a random variable that indicates the first time $t > 0$ that the MC hits $\mathcal{I} \subseteq \mathcal{X}$, i.e.,*

$$T_{\mathcal{I}}^{+} := \min\{t > 0 : x(t) \in \mathcal{I}\} \tag{A.11}$$

Note that the hitting time $T_{\mathcal{I}}$ and return time $T_{\mathcal{I}}^{+}$ coincide if $x(0) \notin \mathcal{I}$. Return time differ from hitting time when $x(0) \in \mathcal{I}$, since $T_{\mathcal{I}} = 0$ and $T_{\mathcal{I}}^{+}$ is the first time such that the random walk returns in $\mathcal{I}$, which satisfies $T_{\mathcal{I}}^{+} > 0$ by construction. We let $p_i(Z)$ denote the probability that an arbitrary event Z occurs given that the MC starts in i, i.e., with $\boldsymbol{\pi}(0) = \delta_i$.

Definition A.13 (Escape probability). *Let $\mathcal{I} \subseteq \mathcal{X}$ be a subset of states. The* escape probability $p_j(T_{\mathcal{I}} < T_j^{+}) \in [0,1]$ *is the probability that the MC starting in j hits the subset $\mathcal{I}$ before returning in j.*

Definition A.14 (Green's function). *Let $G \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ denote the Green's function associated to P, defined as*

$$G := \sum_{t=0}^{\infty} P^t, \tag{A.12}$$

whose element G_{ij} indicates the expected number of times that the random walk starting in i hits j,

We next see that Markov chains are deeply connected to resistor networks, which are the subject of Section A.2. Throughout the dissertation we shall also handle resistor networks with an infinite countable set of nodes. To this aim, we here introduce some notions related to Markov chains with an infinite countable state set. Let us start with the following definition.

Definition A.15 (Recurrent and transient state). *A state i of a Markov chain is said to be* recurrent *if $p_i(T_i^{+} < +\infty) = 1$, otherwise it is called* transient.

Lemma A.2 ([94]). *A state i of a Markov chain is recurrent if and only if $G_{ii} = +\infty$.*

Proof. Let v_i denote the number of visits in i starting from i. By construction,

$$E[v_i] = G_{ii}. \tag{A.13}$$

Also, note that since the Markov chain is time-homogeneous,

$$\mathbf{P}\{v_i > r+1\} = \mathbf{P}\{v_i > r\} \cdot p_i(T_i^+ < +\infty\} \implies \mathbf{P}\{v_i > r\} = \left(p_i(T_i^+ < +\infty)\right)^{r+1}.$$

If i is a recurrent state, then $p_i(T_i^+ < +\infty) = 1$ and $\lim_{r\to+\infty} \mathbf{P}\{v_i > r\} = 1$, yielding $G_{ii} = E[v_i] = +\infty$. Conversely, if i is transient, then $p_i(T_i^+ < +\infty) < 1$, implying

$$G_{ii} = \sum_{r=0}^{\infty} r\left(p_i(T_i^+ < +\infty)\right)^r \left(1 - p_i(T_i^+ < +\infty)\right) = \frac{p_i(T_i^+ < +\infty)}{1 - p_i(T_i^+ < +\infty)} < +\infty,$$

concluding the proof. □

Proposition A.2 ([104]). *In irreducible Markov chains, either all the states are transient or all the states are recurrent.*

Proof. Let us assume that i is a transient state. Since the MC is irreducible, for every $j \in \mathcal{N}$ there exist m and n such that $(P^m)_{ij} > 0$, $(P^n)_{ij} > 0$, and

$$(P^{m+r+n})_{ii} \geq (P^m)_{ij}(P^r)_{jj}(P^n)_{ji}, \tag{A.14}$$

yielding

$$G_{jj} = \sum_{r=0}^{\infty} (P^r)_{jj} \leq \frac{1}{(P^m)_{ij}(P^n)_{ji}} \sum_{r=0}^{\infty} (P^{m+r+n})_{ii} \leq \frac{G_{ii}}{(P^m)_{ij}(P^n)_{ji}} < +\infty, \tag{A.15}$$

which implies that every state j is transient. □

Note that due the previous results, all the states of irreducible Markov chains with finite state set are recurrent. Proposition A.2 allows for a classification of Markov chains, instead of single states, as recurrent or transient.

Definition A.16 (Recurrent Markov chain). *An irreducible Markov chain is recurrent if, for every starting state, it visits its starting state infinitely often with probability one, otherwise it is transient.*

We now explore the connections between Markov chains and networks. In particular, we define and analyse random walks on resistor networks.

Definition A.17. *Given a resistor network $\mathcal{G} = (\mathcal{N}, \mathcal{L}, W)$, a random walk over the network is a MC with state space $\mathcal{X} = \mathcal{N}$ and transition matrix $P = D^{-1}W$.*

We here restrict our analysis to irreducible reversible random walks. As shown in the next proposition, the assumption of reversibility is without loss of generality in resistor networks.

Proposition A.3. *Random walks over resistor networks are reversible with invariant distribution $\boldsymbol{\pi} = \mathbf{w}/w$.*

Proof. We give a constructive proof. First, notice that the elements of $\mathbf{w}/w$ sum to 1 by construction. Moreover, since $W_{ij} = W_{ji}$, it holds:

$$\pi_i P_{ij} = \frac{w_i}{w} P_{ij} = \frac{W_{ij}}{w} = \frac{W_{ji}}{w} = \frac{w_j}{w} P_{ji} = \pi_j P_{ji}, \tag{A.16}$$

concluding the proof. □

Consider a network, and let $\mathcal{K} \subset \mathcal{N}$ be a subset of nodes, with cardinality N_k. Given a random walk on the network, one can define the random walk killed in $\mathcal{K}$, whose transition matrix is ${}_{\mathcal{K}}P \in \mathbb{R}^{(\mathcal{N} \setminus \mathcal{K}) \times (\mathcal{N} \setminus \mathcal{K})}$ by removing from P the rows and the columns referring to nodes $k \in \mathcal{K}$. Note that ${}_{\mathcal{K}}P$ is substochastic because of strong connectedness of the network, which in turn implies the existence of at least a node $j \in \mathcal{N} \setminus \mathcal{K}$ and a node $k \in \mathcal{K}$ such that $P_{jk}, P_{kj} \neq 0$. The *killed random walk* can be thought of as a transient random walk in which a cemetery is created in nodes $\mathcal{K}$, i.e., every time the random walk hits $\mathcal{K}$ it gets absorbed. Note that an invariant distribution for ${}_{\mathcal{K}}P$ does not exist. Indeed, the unique distribution satisfying $\boldsymbol{\pi} = ({}_{\mathcal{K}}P)^T \boldsymbol{\pi}$ is $\boldsymbol{\pi} = \mathbf{0}$, which however does not sum to one and corresponds to the case where all the mass has been absorbed in $\mathcal{K}$. Analogously to standard Markov chains, we define the Green's function ${}_{\mathcal{K}}G \in \mathbb{R}_+^{(\mathcal{N} \setminus \mathcal{K}) \times (\mathcal{N} \setminus \mathcal{K})}$ of the killed random walk ${}_{\mathcal{K}}P$ by

$${}_{\mathcal{K}}G := \sum_{t=0}^{\infty} ({}_{\mathcal{K}}P)^t = (\mathbf{I} - {}_{\mathcal{K}}P)^{-1}. \tag{A.17}$$

The last inequality follows from the fact that ${}_{\mathcal{K}}P$ is substochastic and irreducible. Hence, it has spectral radius $\rho < 1$ and $(\mathbf{I} - {}_{\mathcal{K}}P)^{-1} = \sum_{t=0}^{\infty} ({}_{\mathcal{K}}P)^t$ (see [90]). Since

$((_{\mathcal{K}}P)^t)_{ij}$ is the probability that the killed random walk starting from i is in j after t steps, $_{\mathcal{K}}G_{ij}$ indicates the expected number of times that the killed random walk visits j starting from i before hitting $\mathcal{K}$. By convention, we assume

$$_{\mathcal{K}}G_{kj} = {}_{\mathcal{K}}G_{jk} = 0, \quad \forall k \in \mathcal{K}, j \in \mathcal{N}. \tag{A.18}$$

An equivalent formulation involves the definition of $_{\mathcal{K}}\hat{P} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, obtained by setting to zero the rows and the columns of P referring to elements in $\mathcal{K}$. Thus, the associated Green's function $_{\mathcal{K}}\hat{G} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ satisfies

$$_{\mathcal{K}}\hat{G}_{kj} = {}_{\mathcal{K}}\hat{G}_{jk} = 0, \quad \forall k \in \mathcal{K}, j \in \mathcal{N}, \tag{A.19}$$

by construction and $_{\mathcal{K}}\hat{G}_{ij} = {}_{\mathcal{K}}G_{ij}$ for every $i, j \in \mathcal{N} \setminus \mathcal{K}$. Both these formulations shall be used within this dissertation. Based on the properties of the associated random walk, we define the notion of of recurrent network, which is of interest when studying the limit of infinite networks.

Definition A.18 (Recurrent network)**.** *A resistor network is called recurrent if the random walk $P = D^{-1}W$ on the network is recurrent.*

Every finite connected network is recurrent. As shown in the next examples, this is not true for infinite networks. An equivalent characterization of infinite recurrent networks, provided in [94, Proposition 21.3] is that in recurrent networks

$$\lim_{d\to+\infty} p_i(T_{\partial_d(i)} < T_j) = 0 \quad \forall i, j \in \mathcal{N}, \tag{A.20}$$

where we recall that $\partial_d(i)$ indicates the set of nodes at distance d from node i. Observe that, to hit any node at distance $d+1$, the random walk starting from i has to hit at least a node at distance d. Hence, the sequence $\left\{p_i(T_{\partial_d(i)} < T_j)\right\}_{d=1}^{+\infty}$ is non-increasing in d and the limit is well defined. Eq. (A.20) means that in recurrent networks the probability that the random walk gets at infinite distance vanishes. In the last part of the section we establish useful connections between resistor networks and random walks over the corresponding network. The next proposition states that the Green's function is harmonic with respect to P and is therefore equivalent to the potential $\mathbf{u}$, apart from a proportionality constant.

Proposition A.4 ([94])**.** *Let $\mathcal{G}_R$ be a resistor network, and let $\mathbf{u}$ be the potential distribution under boundary conditions $u_a = 1$, $u_b = 0$. Consider the Green's function* *of the associated random walk killed in b. The function $k \to {}_bG_{ka}$ is harmonic with respect to P for every $a,b \in \mathcal{N}$. Furthermore,*

$$u_k = \frac{{}_bG_{ka}}{{}_bG_{aa}}, \quad \forall k \in \mathcal{N}. \tag{A.21}$$

Proof. Note that ${}_bG_{ka}$ is the expected number of visits in a starting from k before hitting b. Conditioning on the first step of the random walk, one gets

$${}_bG_{ka} = \sum_j {}_bG_{ja}P_{kj}, \tag{A.22}$$

which implies that the function $k \to {}_bG_{ka}$ is harmonic with respect to P, and thus also $k \to {}_bG_{ka}/{}_bG_{aa}$. Since also $\mathbf{u}$ is harmonic with respect to P, it remains only to prove that the boundary conditions are equivalent, and then use Proposition A.1 to establish the equivalence between the two quantities. Since

$$u_a = 1 = \frac{{}_bG_{aa}}{{}_bG_{aa}}, \quad u_b = 0 = \frac{{}_bG_{ba}}{{}_bG_{aa}},$$

boundary conditions are equivalent, concluding the proof. □

Many quantities of interest related to dynamics over networks are harmonic, e.g., escape probabilities in random walks or asymptotic equilibria of DeGroot dynamics in presence of stubborn agents [105]. In particular, by Proposition A.4 and Proposition A.1, one can prove that potential distribution and the Green's function ${}_bG_{\cdot a}$ are proportional to asymptotic opinions of agents under DeGroot dynamics when agents a and b are stubborn with opinion 1 and 0 respectively. These parallelisms may be exploited to borrow results from other fields to analyse resistor networks. Another relation of interest between effective resistance between a pair of nodes in a resistor networks and escape probabilities and Green's function on the correspondent network is provided in the next proposition.

Proposition A.5 ([106]). *Let $\mathcal{G}_R$ be a resistor network and let $P = D^{-1}W$ be the random walk on it. Then, for an arbitrary node k, and for any pair of distinct nodes (a,b), it holds*

$$\frac{{}_kG_{aa} - {}_kG_{ba}}{D_{aa}} + \frac{{}_kG_{bb} - {}_kG_{ab}}{D_{bb}} = \frac{{}_bG_{aa}}{D_{aa}} = \frac{1}{D_{aa}p_a(T_b < T_a^+)} = r_{ab}, \tag{A.23}$$

Example A.3 (Grids)**.** *We consider an infinite 2d square grid of resistors, and assign to every pair of adjacent nodes a unitary conductance. Using symmetry arguments, it can be proved that the effective resistance between an arbitrary pair of adjacent nodes is* $1/2$ *[107]. Moreover, one can show that the infinite 2d square grid is recurrent. The proof exploits the analogy between escape probability and effective resistance established in Proposition A.5. In particular, it can be proved that the network is transient if and only if the effective resistance between an arbitrary node* n *and the set* $\partial_d(n)$ *diverges in the limit of infinite* d*. For more details, see [100, Sections 2.1.6 and 2.2.3]. Interestingly, one can also prove that in contrast with the 2-dimensional grid,* d*-grids with* $d \geq 3$ *are transient. The proof of this statement may be found in [94, Example 21.9].*

Appendix B

Continuous-time dynamical systems

We first introduce the notion of continuous-time dynamical system. Then, we define fixed points of continuous-time dynamical systems and give some definitions on stability of fixed points. Afterwards, we state Lyapunov's and LaSalle's results. Finally, we focus on contractive systems and provide the proof of Proposition 3.2, which establishes a sufficient condition under which a continuous-time dynamical system is contractive.

A *continuous-time dynamical system* is a pair $(\mathcal{X}, g)$, where $\mathcal{X} \subseteq \mathbb{R}^m$, and $g : \mathcal{X} \to \mathbb{R}^m$ is a vector field of class C^r $(r \geq 1)$. Given a continuous-time dynamical system and an initial condition $\mathbf{x}_0$, a *trajectory* of the system is an application $t \to \mathbf{x}(t)$ such that

$$\dot{\mathbf{x}}(t) = g(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{B.1}$$

The assumptions on g ensure the uniqueness of the trajectory for every initial condition. A point $\mathbf{x}^*$ such that $g(\mathbf{x}^*) = \mathbf{0}$ is called a *fixed point* of the system. Note by (B.1) that if $\mathbf{x}_0 = \mathbf{x}^*$, then $\mathbf{x}(t) = \mathbf{x}^*$ for every t. Although the expression *fixed point* is more commonly used for discrete-time systems and *equilibrium point* for continuous-time dynamical systems, we preferred to use this notation to distinguish equilibria of continuous-time dynamical systems from Wardrop equilibria of routing games. We now provide some definitions concerning stability of fixed points.

Definition B.1 (Stable fixed point)**.** *A fixed point* $\mathbf{x}^*$ *is* stable*, if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every initial condition satisfying* $||\mathbf{x}_0 - \mathbf{x}^*|| < \delta$*, then the corresponding trajectory satisfies* $||\mathbf{x}(t) - \mathbf{x}^*|| < \varepsilon$ *for every* t*.*

Thus, a fixed point is stable if any trajectory that starts arbitrarily close to the fixed point remains close to the fixed point. The notion of asymptotic stability additionally requires that the trajectory converges to the fixed point.

Definition B.2 (Asymptotically stable fixed point)**.** *A fixed point* $\mathbf{x}^*$ *is* asymptotically stable *if it is stable and if there exists* $\delta > 0$ *such that for any initial condition satisfying* $||\mathbf{x}_0 - \mathbf{x}^*|| < \delta$ *the corresponding trajectory satisfies*

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^*. \tag{B.2}$$

The following notion provides a tractable tool to establish asymptotic stability of fixed points.

Definition B.3 (Linearly stable fixed point)**.** *Let* $\mathbf{x}^*$ *be a fixed point of the system* $(\mathcal{X}, g)$*, and let* $J|_{\mathbf{x}^*}$ *the Jacobian of g computed in* $\mathbf{x}^*$*. Then,* $\mathbf{x}^*$ *is* linearly stable *if all the eigenvalues of* $J|_{\mathbf{x}^*}$ *have negative real part.*

Proposition B.1. *Linearly stable fixed points are asymptotically stable.*

Proof. See [108, Theorem 3.7]. □

The following definitions concern global stability properties.

Definition B.4 (Globally asymptotically stable fixed point)**.** *Let* $\mathbf{x}^*$ *be an asymptotically stable fixed point of a continuous-time dynamical system* $(\mathcal{X}, g)$*, and let* $\mathcal{C} \subseteq \mathcal{X}$ *be the set of points such that for every* $\mathbf{x}_0 \in \mathcal{C}$

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^*. \tag{B.3}$$

If $\mathcal{C} = \mathcal{X}$*,* $\mathbf{x}^*$ *is a* globally asymptotically stable *fixed point.*

Definition B.5 (Globally exponentially stable fixed point)**.** *Let* $\mathbf{x}^*$ *be a globally asymptotically stable fixed point of a continuous-time dynamical system* $(\mathcal{X}, g)$*. Given a norm* $||\cdot||$*,* $\mathbf{x}^*$ *is* globally exponentially stable *with rate* $c > 0$ *if, for every* $\mathbf{x}_0$*, there exists* $a > 0$ *such that*

$$||\mathbf{x}(t) - \mathbf{x}^*|| \leq a||\mathbf{x}_0 - \mathbf{x}^*||e^{-ct}. \tag{B.4}$$

We now provide two fundamental results on asymptotic behaviour of continuous-time dynamical systems.

Proposition B.2 (Lyapunov global stability criterion)**.** *Consider a continuous-time dynamical system* $(\mathcal{X}, g)$*. Let* $\mathbf{x}^*$ *be a fixed point, and let* $v : \mathcal{X} \to \mathbb{R}$ *satisfy* $v(\mathbf{x}^*) = 0$*,* $v(\mathbf{x}) > 0$ *for every* $\mathbf{x} \neq \mathbf{x}^*$*, and*

$$\dot{v}(\mathbf{x}(t)) = \sum_i \frac{\partial v}{\partial x_i} g_i(\mathbf{x}) < 0. \tag{B.5}$$

If $\mathcal{X}$ *is unbounded, assume additionally that* v *is radially unbounded. Then,* $\mathbf{x}^*$ *is a globally asymptotically stable fixed point.*

Analogous criteria for local stability are established in the literature. The function v is called *(global) Lyapunov function*. Notice that the derivative of v must be strictly negative. LaSalle's invariance principle instead states a convergence result that holds also for non-strict Lyapunov functions. Before establishing LaSalle's invariance principle, we give the following definitions.

Definition B.6 (Invariant set)**.** *Consider a continuous-time dynamical system* $(\mathcal{X}, g)$*. A subset* $\mathcal{S} \subseteq \mathcal{X}$ *is g-invariant if, under the continuous-time dynamical system* $(\mathcal{X}, g)$*,* $\mathbf{x}_0 \in \mathcal{S}$ *implies* $\mathbf{x}(t) \in \mathcal{S}$ *for every* $t \geq 0$*.*

Definition B.7 (Convergence to set)**.** *We say that* $x(t)$ *approaches a set* $\mathcal{S}$ *as* $t \to +\infty$*, indicated by* $x(t) \xrightarrow{t\to+\infty} \mathcal{S}$*, if*

$$\lim_{t\to+\infty} \left(\inf_{s\in\mathcal{S}} ||x(t) - s|| \right) = 0. \tag{B.6}$$

Definition B.8. *Let* $\mathcal{X}$ *be a compact space and* $(\mathcal{X}_n)_n$ *be a sequence of compact sets. We say that* $\lim_{n\to+\infty} \mathcal{X}_n = \mathcal{X}$ *if the following two conditions hold:*

1. *for every* $x \in \mathcal{X}$*, there exists a sequence* $(x_n)_n$ *such that* $x_n \in \mathcal{X}_n$ *for every* n *and* $\lim_{n\to+\infty} x_n = x$*;*
2. *for every converging sequence* $(x_n)_n$*, with* $x_n \in \mathcal{X}_n$ *for every* n*,* $\lim_{n\to+\infty} x_n = x \in \mathcal{X}$*.*

Proposition B.3 (LaSalle's invariance principle [109])**.** *Consider a continuous-time dynamical system* $(\mathcal{X}, g)$*, and let* $\mathcal{S} \subseteq \mathcal{X}$ *be g-invariant. Let* $v(\mathbf{x}(t))$ *a Lyapunov function in* $\mathcal{S}$*, i.e.,* $\dot{v}(\mathbf{x}(t)) \leq 0$ *for every* $\mathbf{x}(t) \in \mathcal{S}$*, and let* $\Omega \subseteq \mathcal{S}$ *the largest set of points* $\mathbf{x} \in \mathcal{S}$ *such that* $\dot{v}(\mathbf{x}) = 0$*. Let* $\mathcal{M} \subseteq \Omega$ *be the largest invariant set contained in* Ω*. Then, for every initial condition* $\mathbf{x}_0 \in \mathcal{S}$*,* $\mathbf{x}(t) \xrightarrow{t\to+\infty} \mathcal{M}$*.*

Definition B.9 (Matrix measure). *Given a norm $||\cdot||$ and a matrix A, the associated matrix measure is*

$$\mu(A) := \lim_{h \to 0^+} \frac{||\mathbf{I} + hA|| - \mathbf{I}}{h}. \tag{B.7}$$

Some of popular matrix measures include:

$$\mu_1(A) = \max_{j \in \{1,\dots,n\}} \left(a_{jj} + \sum_{i=1, i \neq j}^{n} |a_{ij}| \right),$$
$$\mu_2(A) = \lambda_{max} \left(\frac{A + A^T}{2} \right),$$
$$\mu_\infty(A) = \max_{i \in \{1,\dots,n\}} \left(a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}| \right).$$

We now focus on contractive systems. To this aim, we first define the notion of infinitesimal contractivity.

Definition B.10 (Infinitesimal contractivity). *Consider a vector field g, and let J denote its Jacobian matrix. The continuous-time dynamical system $(\mathcal{X}, g)$ is infinitesimally contracting on a set $\mathcal{C} \subseteq \mathcal{X}$ if there exist a norm $||\cdot||$ with associated measure μ, and a constant $c > 0$ such that*

$$\mu(J(\mathbf{x})) \leq -c \quad \textit{for all } \mathbf{x} \in \mathcal{C}. \tag{B.8}$$

We can now prove Proposition 3.2, which states that infinitesimal contractivity implies the existence of a globally exponentially stable fixed point.

Proof of Proposition 3.2

For simplicity of notation we omit the dependence on t. By definition of the l_1-norm and the linearity of the derivative, we get

$$\begin{aligned}\frac{d}{dt}\|\mathbf{x}-\mathbf{y}\|_1 =&\frac{d}{dt}\sum_i |x_i-y_i| = \sum_i \frac{d}{dt}|x_i-y_i| \\ =&\sum_i \operatorname{sign}(x_i-y_i)(\dot{x}_i-\dot{y}_i) \\ =&\sum_i \operatorname{sign}(x_i-y_i)(g_i(\mathbf{x})-g_i(\mathbf{y})) \\ =&\sum_i \operatorname{sign}(x_i-y_i)(g_i(\mathbf{y}+\mathbf{h})-g_i(\mathbf{y})),\end{aligned} \tag{B.9}$$

where $\mathbf{x}=\mathbf{y}+\mathbf{h}$. From

$$\begin{aligned}g_i(\mathbf{y}+\mathbf{h})-g_i(\mathbf{y}) &= \int_0^1 \frac{dg_i(\mathbf{y}+\tau\mathbf{h})}{d\tau}d\tau \\ &= \int_0^1 \sum_j \frac{\partial g_i}{\partial y_j}h_j d\tau,\end{aligned}$$

thus (B.9) is equal to

$$\int_0^1 \sum_i \operatorname{sign}(h_i)\sum_j \frac{\partial g_i}{\partial z_j}h_j d\tau.$$

It then holds

$$\begin{aligned}\sum_i \operatorname{sign}(h_i)\sum_j \frac{\partial g_i}{\partial y_j}h_j &= \sum_i \left(\sum_{j\neq i}\frac{\partial g_i}{\partial y_j}h_j \operatorname{sign}(h_i)+\frac{\partial g_i}{\partial y_i}|h_i|\right) \\ &\leq \sum_i \left(\sum_{j\neq i}\left|\frac{\partial g_i}{\partial y_j}\right||h_j|+\frac{\partial g_i}{\partial y_i}|h_i|\right) \\ &= \sum_j\sum_{i\neq j}\left|\frac{\partial g_i}{\partial y_j}\right||h_j|+\sum_j \frac{\partial g_j}{\partial y_j}|h_j| \\ &= \sum_j |h_j|\left(\sum_{i\neq j}\left|\frac{\partial g_i}{\partial y_j}\right|+\frac{\partial g_j}{\partial y_j}\right) \\ &\leq ||\mathbf{h}||_1\mu_1(J) = ||\mathbf{x}-\mathbf{y}||_1\mu_1(J).\end{aligned}$$

Plugging this in (B.9), one gets

$$\frac{d}{dt}\|\mathbf{x}-\mathbf{y}\|_1 \leq ||\mathbf{x}-\mathbf{y}||_1\mu_1(J) \leq -c||\mathbf{x}-\mathbf{y}||_1, \tag{B.10}$$

which implies (3.30). For the point 2), see [81, Theorem 13].

We finally provide the definition of Metzler matrix.

Definition B.11 (Metzler)**.** *A matrix $A \in \mathbb{R}^{n\times n}$ is Metzler if all the non-diagonal elements are non-negative, i.e., for every $i \neq j$, $A_{ij} \geq 0$.*

Appendix C

Logit dynamics in potential games

We here prove a theoretical result on the asymptotic behaviour of the logit dynamics in potential games. Such a result is not original (see [17, Theorem 7.1.4]), however we here propose an alternative proof for the statement. Although the result holds for arbitrary potential games, we keep the notation of routing games coherently with the rest of the dissertation.

Proposition C.1. *Consider a potential game with potential $V(\mathbf{z})$, and consider the logit(η) defined in* (3.13). *Let*

$$V_\eta(\mathbf{z}) := V(\mathbf{z}) - \frac{1}{\eta} H(\mathbf{z}), \tag{C.1}$$

and

$$\Omega_\eta := \{\mathbf{z} \in \mathcal{Z} : \tau^p \Theta_i^p(\mathbf{z}, \eta) = z_i^p, \ \forall p \in \mathcal{P}, i \in \mathcal{R}\}, \tag{C.2}$$

be the set of fixed points of logit(η), where $\Theta_i^p(\mathbf{z}, \eta)$ are the interaction kernels defined in (3.12). *Then*

1. *all points in Ω_η are stationary points of V_η in $\mathcal{Z}$;*
2. $\frac{d}{dt} V_\eta(\mathbf{z}(t)) = -\frac{1}{\eta} \sum_{p \in \mathcal{P}} \tau^p \sum_{i \in \mathcal{R}} \left(\ln \Theta_i^p(\mathbf{z}) - \ln \left(\frac{z_i^p}{\tau^p} \right) \right) \left(\Theta_i^p(\mathbf{z}) - \frac{z_i^p}{\tau^p} \right) \leq 0,$
3. *For every initial condition $\mathbf{z}(0)$, $\mathbf{z}(t) \xrightarrow{t \to +\infty} \Omega_\eta$.*

Proof. *1)* To prove point 1 we show that if $\mathbf{z} \in \Omega_\eta$, then for every $i, j \in \mathcal{R}$ and $p \in \mathcal{P}$

$$\frac{\partial V_\eta}{\partial z_i^p} - \frac{\partial V_\eta}{\partial z_j^p} = 0. \tag{C.3}$$

To this aim, note that

$$\begin{aligned}
\frac{\partial V_\eta}{\partial z_i^p} - \frac{\partial V_\eta}{\partial z_j^p} &= c_i^p(\mathbf{z}) - c_j^p(\mathbf{z}) + \frac{1}{\eta}\left(1 + \ln\frac{z_i^p}{\tau^p} - 1 - \ln\frac{z_j^p}{\tau^p}\right) \\
&= \frac{1}{\eta}(\ln\Theta_j^p(\mathbf{z}) - \ln\Theta_i^p(\mathbf{z})) + \frac{1}{\eta}\left(\ln\frac{z_i^p}{\tau^p} - \ln\frac{z_j^p}{\tau^p}\right) \\
&= \frac{1}{\eta}\left(\ln\Theta_j^p(\mathbf{z}) - \ln\frac{z_j^p}{\tau^p}\right) - \frac{1}{\eta}\left(\ln\Theta_i^p(\mathbf{z}) - \ln\frac{z_i^p}{\tau^p}\right) = 0.
\end{aligned} \tag{C.4}$$

where the first equivalence follows from the definition of potential, the second one from the explicit form of the interaction kernels, and the last one from $\mathbf{z} \in \Omega_\eta$.

2) Since the logit dynamics is exact target, from Lemma 3.1 it holds for every population p

$$\sum_{i\in\mathcal{R}} (\tau^p\Theta_i^p(\mathbf{z}) - z_i^p) = 0. \tag{C.5}$$

Using repeatedly (C.5),

$$\begin{aligned}
\frac{d}{dt}V(\mathbf{z}(t)) &= \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}} \frac{\partial V}{\partial z_i^p}(\tau^p\Theta_i^p(\mathbf{z}) - z_i^p) \\
&= \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}} \left(\frac{\partial V}{\partial z_i^p} - \frac{\partial V}{\partial z_j^p}\right)(\tau^p\Theta_i^p(\mathbf{z}) - z_i^p) \\
&= \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}} \left(c_i^p - c_j^p\right)(\tau^p\Theta_i^p(\mathbf{z}) - z_i^p) \\
&= \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}} c_i^p(\tau^p\Theta_i^p(\mathbf{z}) - z_i^p) \\
&= -\frac{1}{\eta}\sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}} \ln\Theta_i^p \cdot (\tau^p\Theta_i^p(\mathbf{z}) - z_i^p),
\end{aligned} \tag{C.6}$$

where the third equivalence follows from the definition of potential, and the last one from the explicit form of the interaction kernels. Moreover,

$$\begin{aligned}\frac{d}{dt}H(\mathbf{z}(t)) &= \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}}\frac{\partial H}{\partial z_i^p}(\tau^p\Theta_i^p(\mathbf{z}) - z_i^p)\\ &= -\sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{R}}\ln\left(\frac{z_i^p}{\tau^p}\right)(\tau^p\Theta_i^p(\mathbf{z}) - z_i^p).\end{aligned} \tag{C.7}$$

Plugging (C.6) and (C.7) in (C.1), we finally get

$$\frac{d}{dt}V_\eta(\mathbf{z}(t)) = -\frac{1}{\eta}\sum_{p\in\mathcal{P}}\tau^p\sum_{i\in\mathcal{R}}\left(\ln\Theta_i^p(\mathbf{z}) - \ln\left(\frac{z_i^p}{\tau^p}\right)\right)\left(\Theta_i^p(\mathbf{z}) - \frac{z_i^p}{\tau^p}\right) \le 0, \tag{C.8}$$

which proves point 2.

3) Since logit dynamics is exact target, $\mathcal{Z}$ is invariant due to Lemma 3.1. Thus, point 3 follows from points 1 and 2, and from LaSalle's invariance principle (see Proposition B.3). □

Appendix D

Double tree network

We prove that the double tree network in Figure D.1 is not recurrent by showing that $p_i(T_i < T_{\partial_d})$ is equivalent to the same quantity computed for the biased random walk. Indeed, let us identify all the nodes in the left (right) tree that are at distance d from i (or j) in a unique node. Then, the probability of going from a node at distance d from i to a node at distance $d+1$ and $d-1$ are $2/3$ and $1/3$, respectively. Hence, the double tree is equivalent to a biased random walk on a line as in Figure D.2. Since the biased random walk is transient (see [94]), also the double tree is transient. We now show that also Term 2 does not vanishes in the limit of infinite distance. Since in the original network and in the cut network there are no paths between i and j except the link l joining i and j (see Fig. D.4 (a) and (b)),

$$r_{ij} = r_{ij}^{U_d} = 1.$$

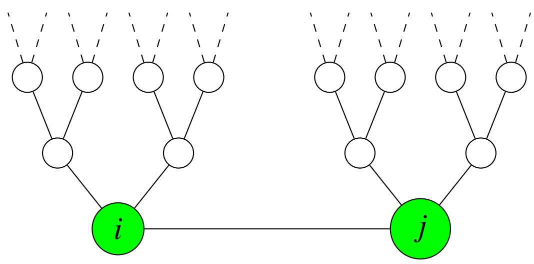

Fig. D.1 The double tree is an infinite non-recurrent network. On this network $\lim_{d\to+\infty} \varepsilon_{ijd} = 1/3$.

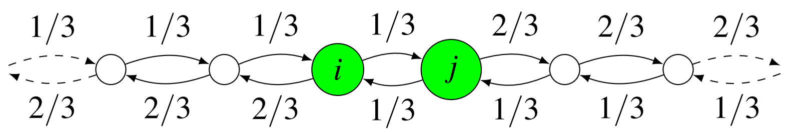

Fig. D.2 The double tree network is equivalent to a biased random walk.

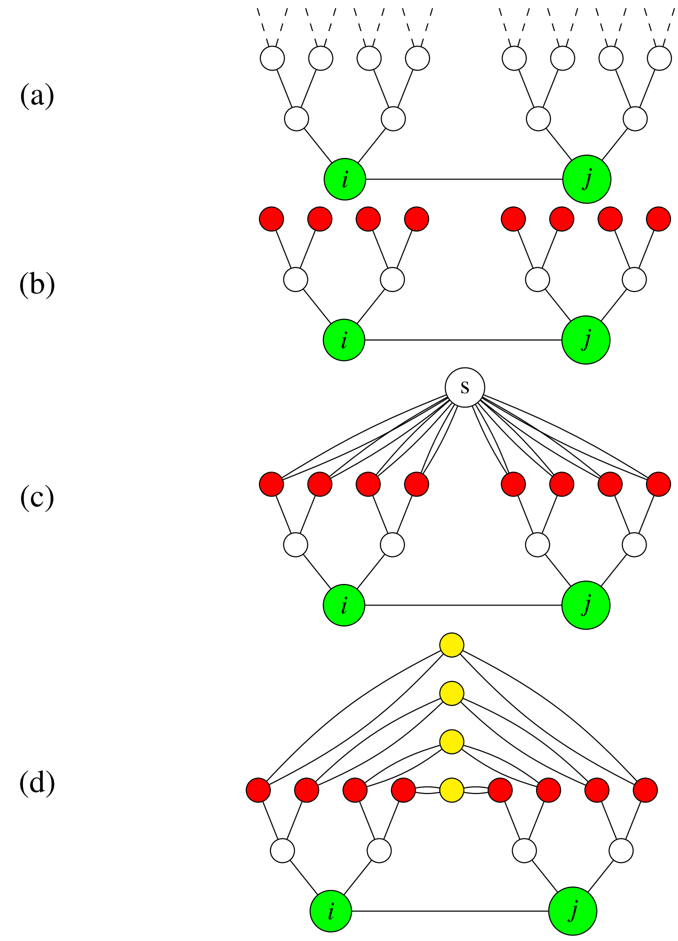

Fig. D.3 From above to below: (a) the double tree network; (b) the cut tree network at distance 2 from $\{i, j\}$; (c) the shorted tree network at distance 2 from $\{i, j\}$; (d) a network equivalent to the shorted one. In red, the nodes at distance 2 from $\{i, j\}$.

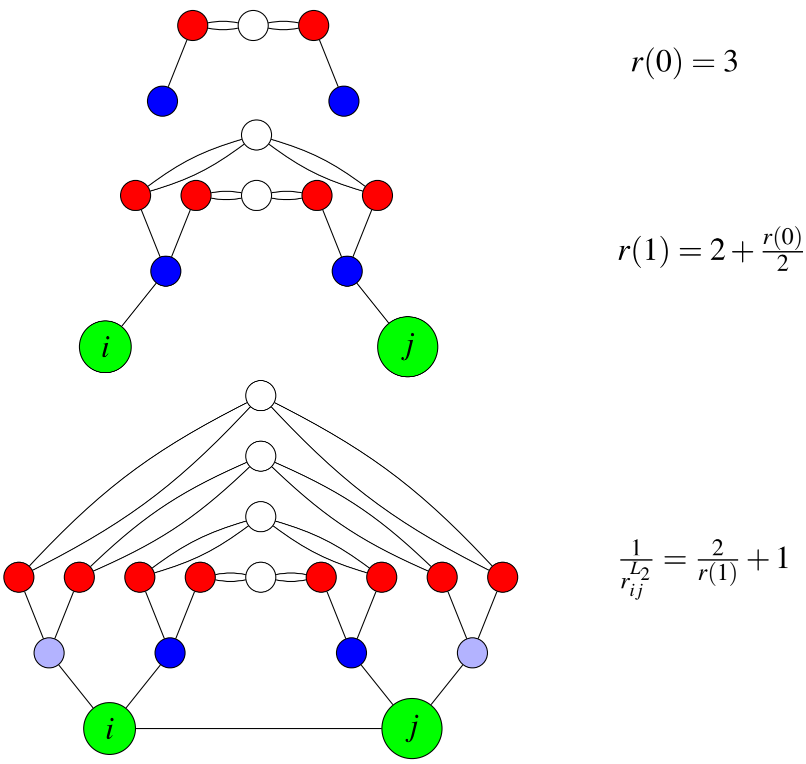

Fig. D.4 The network in Fig. D.3(d) is series-parallel. Then, it can be obtained by recursively making parallel and series compositions of series-parallel network.

Computing $r_{ij}^{L_d}$ is more involved. First, referring to Figure D.3, we note that, because of the symmetry of the network, the effective resistance between i and j in the shorted network (c), which is $r_{ij}^{L_d}$, is equivalent to the effective resistance in (d). Indeed, if we set potential $u_i = 1$ and $u_j = 0$, because of symmetry every yellow node has potential $1/2$. Thus, adding infinite conductance between all of them, i.e., shorting them, does not affect the electrical current in the network (the procedure of shorting nodes with same potential is also known in the literature as *gluing*, see [94]), and therefore the effective resistance. The network (d) is series-parallel, so that the effective resistance can be computed iteratively. Specifically, we refer to Figure D.4 to explain the recursion that leads to $r_{ij}^{L_d}$. From top to bottom, it note that the first network has effective resistance between the two blue nodes equal to 3. The second network is the parallel composition of two of these, in series with two single links. This procedure is iteratively repeated $d-1$ times (in Figure D.4 only once, since $d = 2$), leading to a network that, composed in parallel with a copy of itself and with a single link, is $\mathcal{G}_{ij}^{L_d}$. Hence, $r_{ij}^{L_d}$ is the result of the following recursion.

$$\begin{cases} r(0) = 3, \\ r(n) = 2 + \frac{r(n-1)}{2}, \quad d > n \geq 1, \\ r_{ij}^{L_d} = (1 + \frac{2}{r(d-1)})^{-1}, \end{cases}$$

which has solution

$$\begin{cases} r(n) = (2^{d+2} - 1)/2^d, \quad d > n \geq 1, \\ r_{ij}^{L_d} = \frac{2^{d+1}-1}{2^{d+1}+2^d-1} \xrightarrow{d \to +\infty} \frac{2}{3}. \end{cases}$$